Schwarz Methods for the Biharmonic Equation

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Abstract

We study classical and optimized Schwarz methods for the biharmonic equation. This equation, which needs two different boundary conditions, is quite different from the classical Laplace equation, and the classical Schwarz method converges much more slowly than in the Laplace case. Through suitable choices of the transmission conditions, as well as their corresponding parameters, we obtain an optimized Schwarz method with a convergence rate that is exactly the same as for optimized Schwarz in the Laplace case. We illustrate our theoretical results with numerical experiments.

Classical Schwarz Method for the Biharmonic Equation

The performance of the classical Schwarz method for fourth order problems has been studied already in the literature. Brenner analyzed a two level additive Schwarz preconditioner in [1] and showed that the condition number is of order $1+(\frac{H}{\delta})^4$ for large overlap and $1+(\frac{H}{\delta})^3$ for small overlap. The FETI method was proposed and studied by Farhat and Mandel in [2], and Mandel, Tezaur and Farhat in [4], where continuity of the transverse displacements is enforced at the substructure corners, and the condition number is $O(1+\log\frac{H}{h})^3$. A non-overlapping Schwarz preconditioner for a discontinuous Galerkin discretization was introduced by Feng and Karakashian in [3], with a condition number estimate $O(1+\frac{H}{h})^3$. A Schwarz waveform relaxation method was introduced by Nourtier-Mazauric and Blayo in [5] for the time dependent problem, with an optimal choice of the parameters in the transmission conditions, illustrated by numerical experiments, but without analysis. The convergence rate for the classical Schwarz method was studied by Shang and He in [6], which is the starting point of our poster. We consider the biharmonic equation

$$\Delta^2 u = f \quad \text{in } \Omega,$$

where $\Omega = R^2$, and the solution u decays at infinity. We assume that Ω is divided into two subdomains $\Omega_1 = (-\infty, x_1)\times\mathbb{R}$ and $\Omega_2 = (x_2, +\infty)\times\mathbb{R}$, where $x_1 \geqslant x_2$. Let Γ_i, $i=1,2$ be the interface at $x = x_i$, and define $L := x_1 - x_2$.

Given an initial approximation u_2^0, the classical Schwarz method computes for $n = 1, 2, \ldots$

$$\begin{aligned} \Delta^2 u_1^n &= f_1 && \text{in } \Omega_1, & \Delta^2 u_2^n &= f_2 && \text{in } \Omega_2,\\ u_1^n &= u_2^{n-1} && \text{on } \Gamma_1, & u_2^n &= u_1^n && \text{on } \Gamma_2,\\ \frac{\partial u_1^n}{\partial n_1} &= \frac{\partial u_2^{n-1}}{\partial n_1} && \text{on } \Gamma_1, & \frac{\partial u_2^n}{\partial n_2} &= \frac{\partial u_1^n}{\partial n_2} && \text{on } \Gamma_2. \end{aligned}$$

Taking a Fourier transform in the y direction with k the Fourier symbol, and assuming that the relevant numerical Fourier frequencies $|k|$ lie in the interval $[k_{min}, k_{max}]$ with $k_{min}, k_{max} > 0$, we obtain by a direct computation (see also [6]):

Proposition 1: If $L > 0$, the convergence factor for the classical Schwarz method with two subdomains applied to the biharmonic equation is given by

$$\rho(L) = (|k|L + \sqrt{|k|^2L^2+1})^2 e^{-2|k|L} = 1 - \frac{1}{3}k^3L^3 + O(L^5) < 1.$$

It is the additional factor $(|k|L + \sqrt{|k|^2L^2+1})^2$ which leads to the substantially worse behavior of the classical Schwarz method for low frequencies compared to the Laplace case, where $\rho_{Laplace}(L) = 1 - 2kL + O(L^2)$. We illustrate this in Table 1 and Figure 1.

L	$h=\frac{1}{16}$	$h=\frac{1}{32}$	$h=\frac{1}{64}$	$h=\frac{1}{128}$
h	853(6)	6469(9)	50906(12)	>200000(14)
$2h$	235(5)	1655(8)	12819(11)	101157(14)
$4h$	53(4)	305(7)	2189(9)	16971(13)

Table 1: Iteration numbers for the classical Schwarz method (in parentheses optimized Schwarz method) for the biharmonic equation with different overlap sizes.

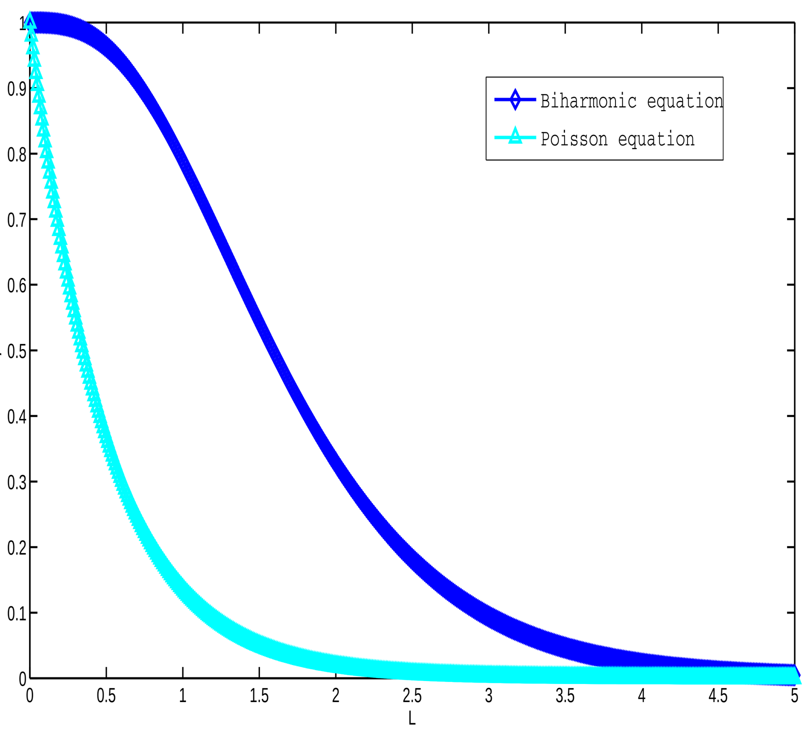

Figure 1: Convergence factors corresponding to an overlap L for the biharmonic and the Laplace equations.

Optimized Schwarz Method for the Biharmonic Equation

An optimized Schwarz method uses different transmission conditions: given an initial approximation u_2^0, the method computes for $n = 1, 2, \ldots$

$$\begin{aligned} &\Delta^2 u_1^n = f_1 && \text{in } \Omega_1,\\ -\begin{bmatrix} -\Delta u_1^n \\ \partial_{n_1}\Delta u_1^n \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} u_1^n \\ \partial_{n_1} u_1^n \end{bmatrix} &= -\begin{bmatrix} -\Delta u_2^{n-1} \\ \partial_{n_1}\Delta u_2^{n-1} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} u_2^{n-1} \\ \partial_{n_1} u_2^{n-1} \end{bmatrix} && \text{on } \Gamma_1,\\ &\Delta^2 u_2^n = f_2 && \text{in } \Omega_2,\\ -\begin{bmatrix} -\Delta u_2^n \\ \partial_{n_2}\Delta u_2^n \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} u_2^n \\ \partial_{n_2} u_2^n \end{bmatrix} &= -\begin{bmatrix} -\Delta u_1^n \\ \partial_{n_2}\Delta u_1^n \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} u_1^n \\ \partial_{n_2} u_1^n \end{bmatrix} && \text{on } \Gamma_2. \end{aligned}$$

Proposition 2: If the elements in the matrix $P = [p_{ij}]$ are chosen in the Fourier domain as

$$\widetilde{P} = \begin{bmatrix} 2|k|^2 & 2|k| \\ 2|k|^3 & 2|k|^2 \end{bmatrix},$$

we obtain an optimal Schwarz method with convergence in two iterations.

The choice in Proposition 2 corresponds to a transparent transmission condition and requires the implementation of a non-local operator.

Proposition 3: With the simple but structural consistent constant choice

$$P_{\text{approx}} = \begin{bmatrix} 2p^2 & 2p \\ 2p^3 & 2p^2 \end{bmatrix}, \quad p \geqslant 0,$$

the convergence factor of the optimized Schwarz method is

$$\rho(L) = \left(\frac{p-|k|}{p+|k|}\right)^2 e^{-2|k|L} < 1,$$

which is the same as for the optimized Schwarz method applied to Laplace's equation. In the numerical experiments, we thus choose $p = h^{-1/2}$ for the nonoverlapping case to get $\rho = 1 - O(\sqrt{h})$, and $p = (\frac{h}{16})^{-1/3}$ for the overlapping case which leads to $\rho = 1 - O(h^{\frac{1}{3}})$.

Proposition 4: For the nonoverlapping case, if the matrix is of the form

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix},$$

where p_{ij}, $i,j = 1,2$ are constants independent of each other, then the optimal choice of the parameters is

$$\begin{aligned} p_{11} = p_{22} &\geqslant 0,\\ p_{12}p_{21} &= p_{11}^2,\\ \frac{p_{21}}{p_{12}} &= k_{min}k_{max}. \end{aligned}$$

The corresponding optimal convergence factor is

$$\rho = \left(\frac{\sqrt{k_{max}} - \sqrt{k_{min}}}{\sqrt{k_{max}} + \sqrt{k_{min}}}\right)^2 < 1.$$

So the choice P_{approx} in Proposition 3 is optimal.

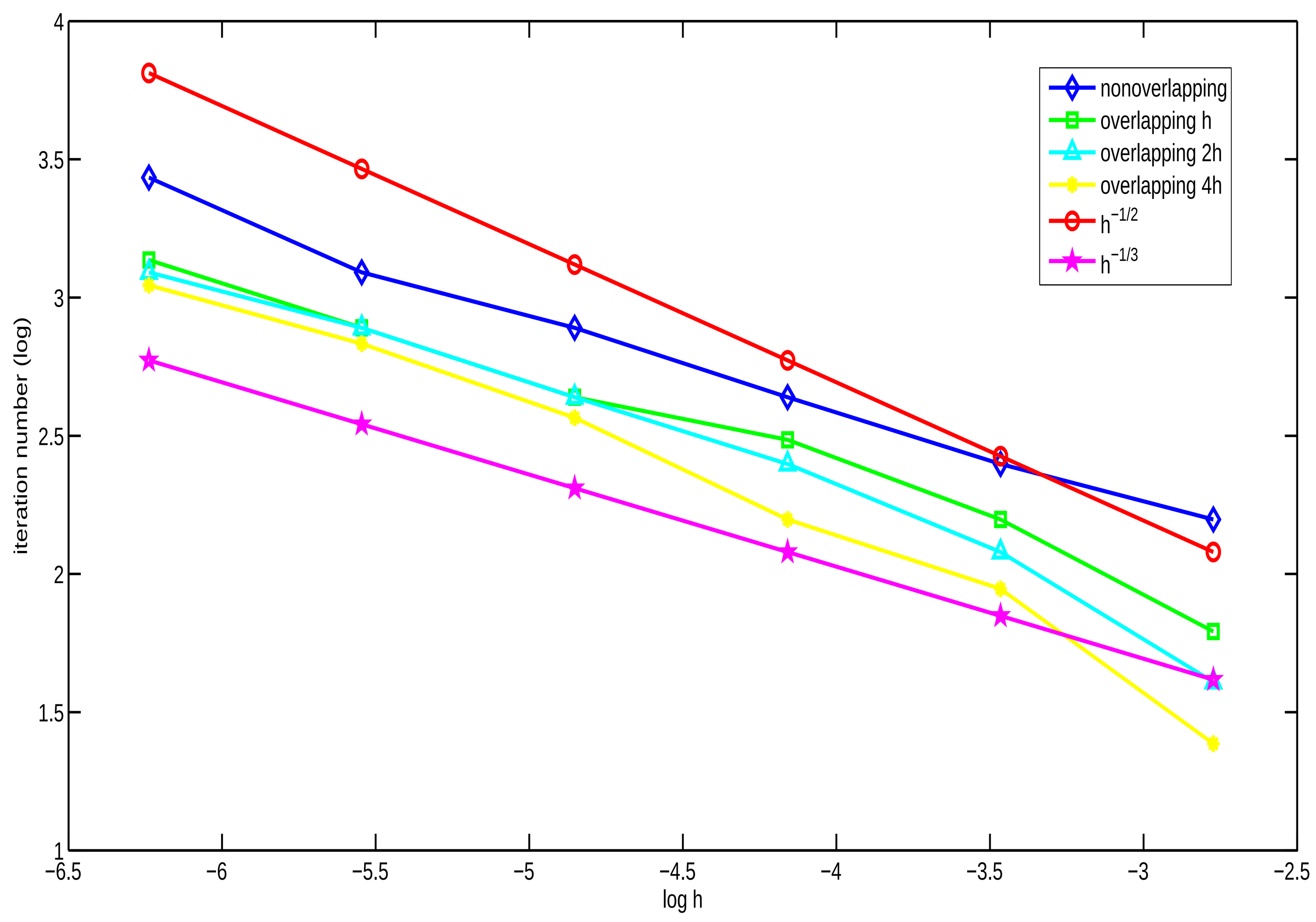

Figure 2: Iteration numbers for the optimized Schwarz method with different overlap sizes.

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