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Shape design problem of waveguide by controlling resonance

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Abstract

We develop a numerical method to design the acoustic waveguide shape which has the filtering property to reduce the amplitude of frequency response in a given target bandwidth. The basic mathematical modeling is given by the acoustic wave equation and the related Helmholtz equation, and we compute complex resonant poles of the wave guide by finite element method with Dirichlet-to-Neumann mapping imposed on the domain boundary between bounded and unbounded domains. We adopt the gradient method to design the desired domain shape using the variational formula for complex resonant eigenvalues with respect to the shape modification of the domain.

●Introduction

★Two typical roles of wave propagation:

1) Energy transportation: sunlight, electric current, seismic wave(ex. earthquake), water wave (ex. tsunami)

2) Information transmission: speech, music, electromagnetic wave (ex. radar, light), underwater acoustic wave(ex. sonar)

★Mathematical description of wave phenomena:

1) Wave equation (as partial differential equation)

2) Evolution equation (as operator theoretical formulation)

★Three important elements in wave propagation:

1) Source or Input (Origin)

2) Filtering or Modulation (with respect to amplitude and phase)

3) Observation or Output (Influence)

★ Characteristic phenomena: Scattering and Resonance

Contents of talk in some details with key words

- Review some analytical and numerical methods for (time-harmonic) wave propagation and radiation problem, i.e. Helmholtz equation
- Application to Wave guide filtering problem for frequency response with a typical application to voice generation
- Characterization of the wave guide via "Frequency response function" defined as the peaks of the frequency response function
- Shape designing of the wave guide via complex Resonance eigenvalues given by the analytic continuation of the frequency response function which determine desirable frequency response
- Sensitivity analysis based on Variational formula of eigenvalue plays an essential role

● Numerical methods for wave propagation problem

★ Mathematical formulation as PDE

Wave equation: $(\frac{\partial^2}{\partial t^2} - c^2\Delta)u(x,t) = f(x,t) \quad \text{in } \Omega \subset R^{n(=1,2,3)},$ c : sound velocity

Assuming time harmonicity of source term f and then u :

$$\downarrow \quad u(x,t) = u(x)e^{-i\omega t}, f(x,t) = f(x)e^{-i\omega t}$$

Helmholtz equation: $(-\Delta - k^2)u(x) = f(x) \quad \text{in } \Omega \quad k = \frac{\omega}{c}$

with outgoing radiation condition (due to causality):

In circular or spherical exterior cases, it is the Sommerfeld radiation condition:

$$\lim_{|x|\to\infty} r^{(n-1)/2}\left\{\frac{\partial u}{\partial r}(x) - iku(x)\right\} = 0$$

★ Review of the results for obstacle scattering problem

Consider the evolution equation with self-adjoint operator H in $L^2(\Omega)$:

$$\frac{d}{dt}u(t) = iHu(t),\ u(0) = u_0 \text{ in } L^2(\Omega),\ R^n \supset \Omega\ ,\ \Omega^c\text{:obstacle}$$

1) Existence of wave operators: $u(t)$ tends to $u_0(t)$ of unperturbed system: $\frac{d}{dt}u_0(t) = iH_0u_0(t),\ u_0(0) = u_{00}$ in $L^2(R^n)$.

The first question we may ask is the existence of wave operators $W_\pm$:

$$W_\pm \equiv s - \lim_{t\to\pm\infty} \exp(-itH)\, J \exp(itH_0), J: L^2(R^n) \to L^2(\Omega)$$

2) Completeness of wave operators: Range(W_+)=Range(W_-).

3) Some properties of scattering operator $S \equiv W_+{}^* W_-$ related to resonances for example.

4) Extending the results to the case of wave equation (see [3]).

References:

[1] Shenk, N. and Thoe, D., Resonant States and Poles of the Scattering Matrix for Perturbations of $-\Delta$ Journal of Mathematical Analysis and Applications, 37, 467-491 (1972),

[2] Kuroda, S. T., Scattering theory for differential operators, III; exterior problems, Spectral Theory and Differential Equations. Springer Berlin Heidelberg, 227-241 (1975),

[3] Kako, T., Scattering theory for abstract differential equations of the second order, J. Fac. Sci., Univ. Tokyo, Sect. IA 19, 377-392 (1972) .

Reduction of the problem in a bounded domain

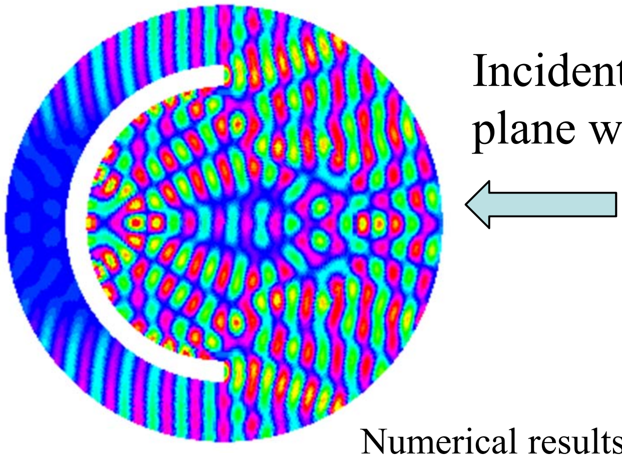
★ Radiation problem for 2D circular exterior case:

u: sound pressure

$$-\Delta u - k^2 u = 0 \quad \text{in} \quad \Omega_R = \Omega \cap B_R$$

$$\frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial\Omega$$

$$\frac{\partial u}{\partial r} = -M(k)u \quad \text{on} \quad \Gamma_R.$$



Numerical results by Dr. H.M. Nasir

where $M \equiv M(D^2)$ called the Dirichlet-to-Neumann mapping, is a function of $D^2 = -\partial^2 / \partial\theta^2$:

$$(M(k)u)(\theta) = -\frac{k}{2\pi} \sum_{n=-\infty}^{\infty} \frac{H^{(1)'}(kR;n)}{H^{(1)}(kR;n)} \int_0^{2\pi} u(R,\phi) e^{in(\theta-\phi)} d\phi$$

$$= -k \frac{H^{(1)'}(kR;\sqrt{D^2})}{H^{(1)}(kR;\sqrt{D^2})} u(R,\theta)$$

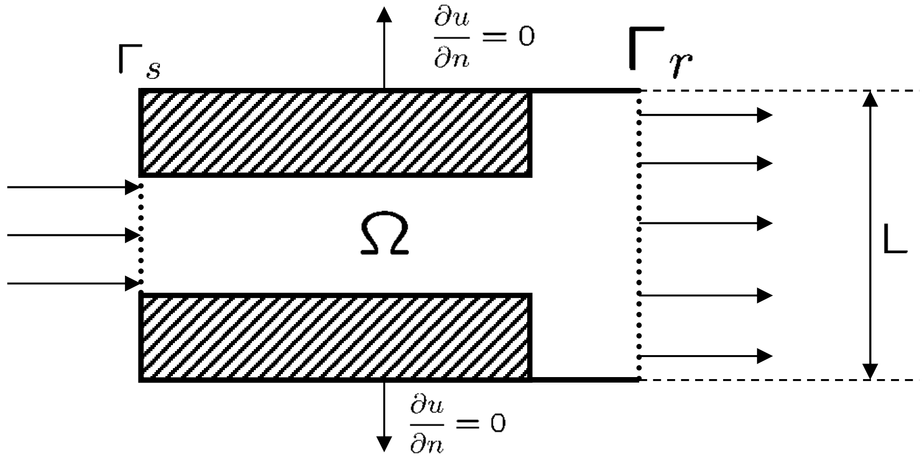
Where $H^{(1)}(x;\nu) := H_\nu^{(1)}(x)$ is the Hankel function of the first kind of order one, and ' denotes the derivative w. r. t. x.

★ Radiation problem for 2D cylindrical exterior case

$$-\Delta u - k^2 u = 0 \quad \text{in } \Omega_i$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega \cap \overline{\Omega_i}$$

$$\frac{\partial u}{\partial n} = -M(k)u \quad \text{on } \Gamma_R.$$



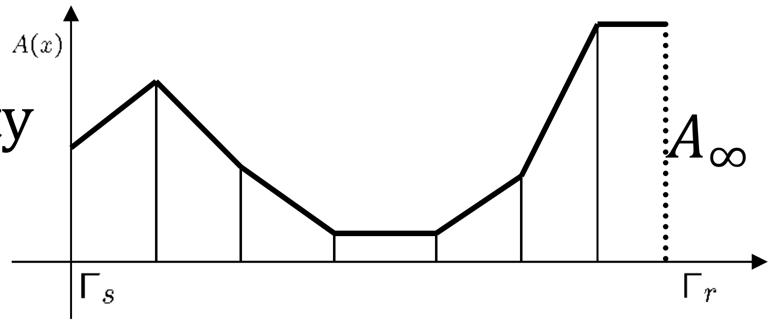
where $M(k) = M(k; D^2)$, a function of $D^2 = -\partial^2 / \partial y^2$.

$$(M(k)u)(L,y) = \sum_{n=0}^{\infty} \zeta_n \int_0^{y_0} u(L,z) c_n(z) dz\, c_n(y), \quad c_n(y) = \begin{cases} \sqrt{\frac{1}{y_0}} & (n=0) \\ \sqrt{\frac{2}{y_0}} \cos(\frac{n\pi}{y_0} y) & (n \geq 1) \end{cases}$$

$$\zeta_n = \begin{cases} i\xi_n, & \xi_n = \left\{k^2 - (\frac{n\pi}{y_0})^2\right\}^{1/2}, \quad 0 \leq n < \frac{y_0}{\pi} k \\ -\eta_n, & \eta_n = \left\{(\frac{n\pi}{y_0})^2 - k^2\right\}^{1/2}, \quad \frac{y_0}{\pi} k \leq n \end{cases}$$

★ 1D-Webster's Model

u: sound pressure, v: volume velocity
$A(x)$: area function,
ρ: density, c: sound velocity



$$\left.\begin{aligned} -\frac{\partial v}{\partial t} &= \frac{A(x)}{\rho}\frac{\partial u}{\partial x}, \\ -\frac{\partial u}{\partial t} &= \frac{\rho c^2}{A(x)}\frac{\partial v}{\partial x}, \end{aligned}\right\} \Rightarrow \frac{\partial^2 u}{\partial t^2} - \frac{c^2}{A(x)}\frac{\partial}{\partial x}\left(A(x)\frac{\partial u}{\partial x}\right) = 0,$$

★Time harmonic stationary reduced wave equation

$$-\frac{1}{A(x)}\frac{\partial}{\partial x}\left(A(x)\frac{\partial u}{\partial x}\right) - k^2 u = 0, \qquad k = \omega / c,$$

$$\frac{du}{dx}(0) = 1, \quad \frac{du}{dx}(L) = iku(L) \equiv -M(k)u(L),$$

Weak formulation and discretization by FEM

Let $H^1(\Omega_R)$ the Sobolev space of order one,

$\gamma : H^1(\Omega_R) \to H^{1/2}(\Gamma_R)$, trace operator on Γ_R

Find $u \in V$ such that

$$a(u,v) + \langle \gamma u, \gamma v \rangle_{M(k)} = (g,v)_{\partial\Omega \backslash \Gamma_R}, \qquad \forall v \in V,$$

where

$$a(u,v) = \int_{\Omega_R} \left[\nabla u \cdot \nabla \overline{v} - k^2 u \overline{v} \right] dxdy, \quad u, v \in V$$

$$\langle p, q \rangle_{M(k)} = \int_0^{2\pi} (M(k)p)(\theta) \overline{q}(\theta) R d\theta, \qquad p, q \in H^{1/2}(\Gamma_R)$$

$$(f, g)_{\partial\Omega} = \int_{\partial\Omega} f \cdot \overline{g} d\sigma, \qquad f, g \in L^2(\partial\Omega).$$

Finite dimensional approximation

Let $V_h \subset V,\ 0 \le h \le h_0$ be a finite dimensional subspace of *V*.

Find $u_h \in V_h$ such that

$$a(u_h, v_h) + \langle \gamma u_h, \gamma v_h \rangle_{M(k)} = (g, v_h)_{\partial\Omega}, \quad \forall v_h \in V_h,$$

Choosing basis $\{\Psi_I\}_{I=1}^N$ in V_h, we have a matrix equation

$$AU + MU = F$$

where $A_{IJ} = a(\Psi_J, \Psi_I), \quad M_{IJ} = \langle \Psi_J, \Psi_I \rangle_{M(k)}$

$U = [U_1, U_2, \ldots, U_N]$ with $u_h = \sum_{K=1}^{N} U_K \Psi_K$

$F = [F_1, F_2 = 0, F_3 = 0, \ldots]$ with $F_I = (\partial u^{\mathrm{inc}} / \partial n, \Psi_I)_{\partial\Omega}$.

There are several results on the convergence of approximation. One method is based on Mikhlin's result ([5]) for compactly perturbed problem using the Fredholm alternative theorem and unique continuation property (see, for example Kako [4]).

● Mathematical modeling and Numerical simulation of wave propagation in wave guide

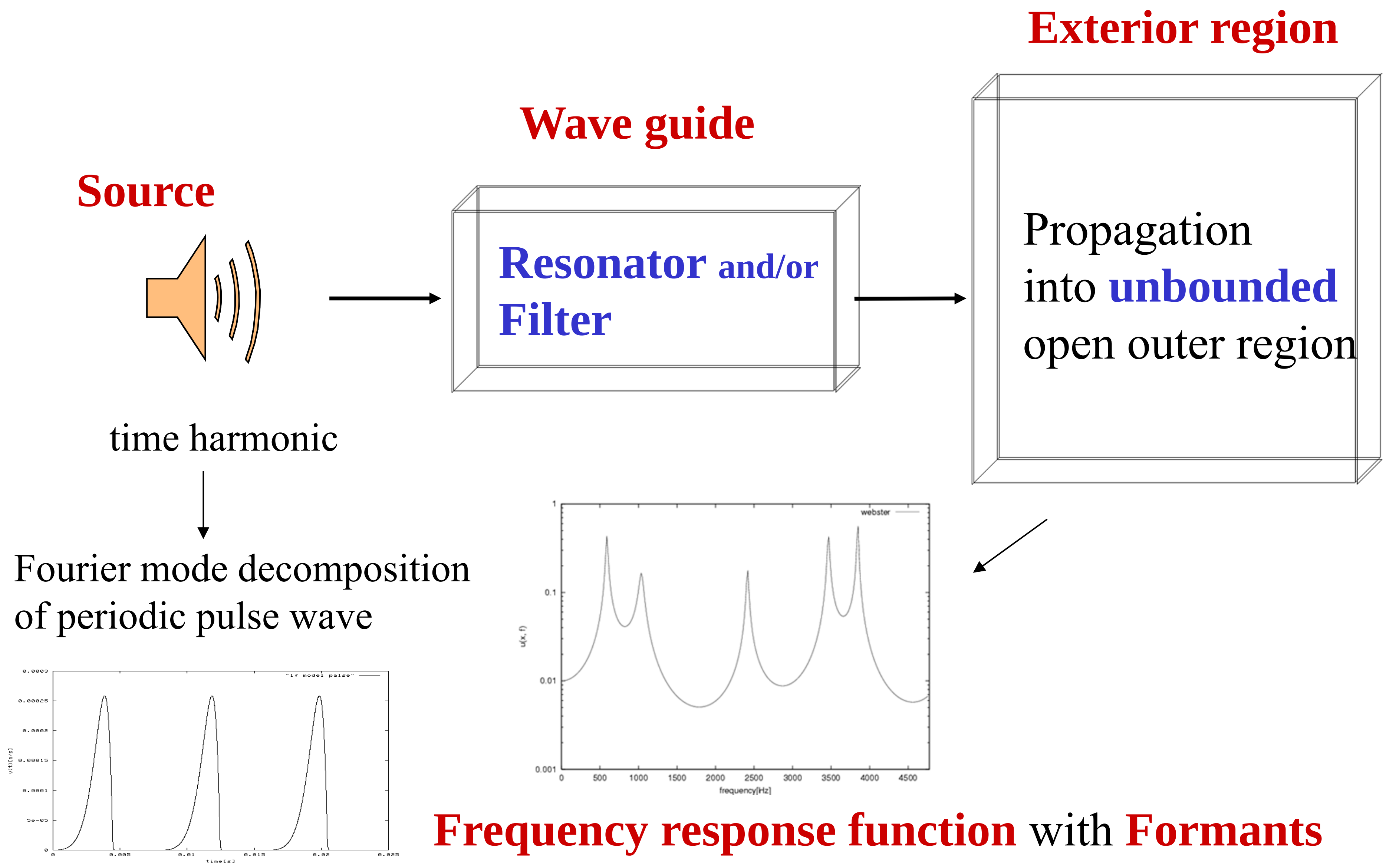
1. Mathematical modeling of wave guide problem

- Wave propagation phenomena in waveguide or in another unbounded region: Wave equation and radiation problem (based on mathematical scattering theory)
- Time harmonic equation : Helmholtz equation and radiation condition at outer boundary or at infinity which is generalized eigenvalue problem related to the continuous spectrum
- Frequency response function and its analytic continuation (resonance phenomena)

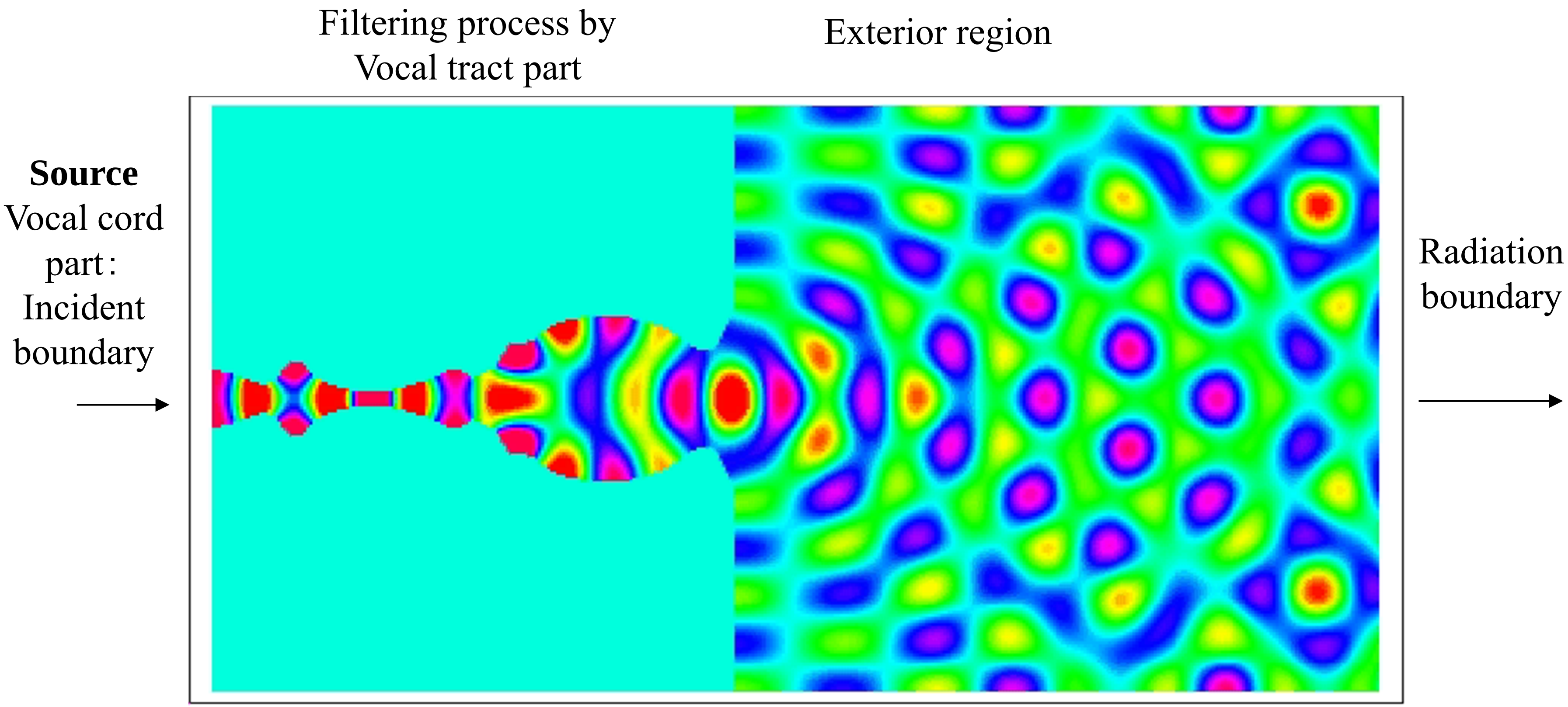
2. Discrete approximation method by Finite Element Method (FEM)

- Reduction to the problem in bounded region via the DtN mapping or its approximation
- Introduction of approximation space and its basis functions
- Construction of approximation equation by projection method (FEM)
- Numerical algorithm and some theoretical considerations

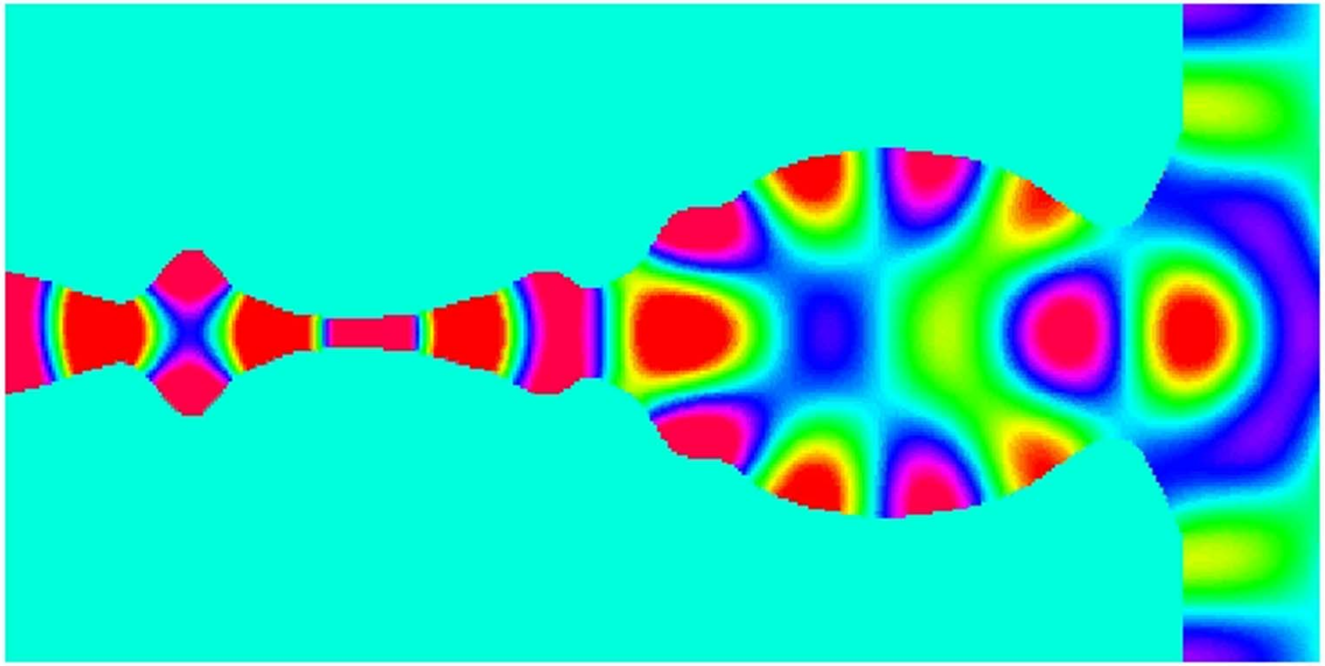
★ Schematic diagram of open wave guide



Numerical examples in voice generation phenomena through vocal tract（ω=70000[Hz], c = 33145[cm/s^2]）

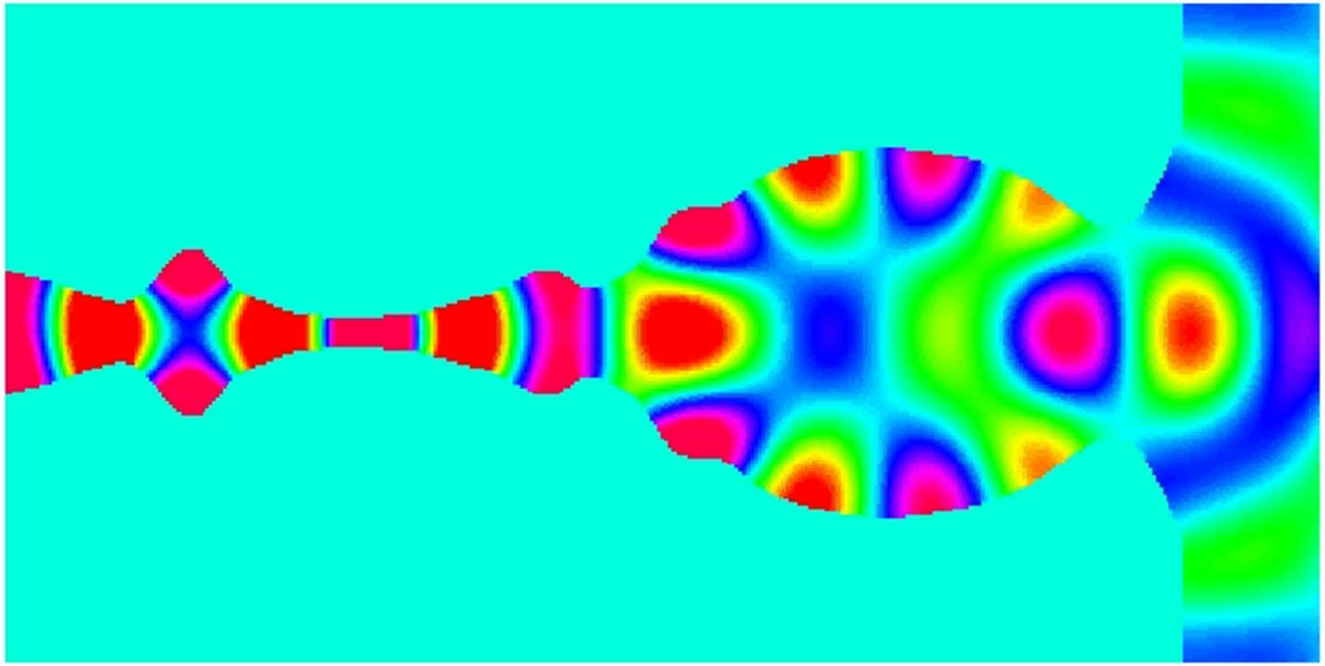


Vocal cord part: Incident boundary



Radiation boundary with plane wave approximation

Vocal cord part: Incident boundary



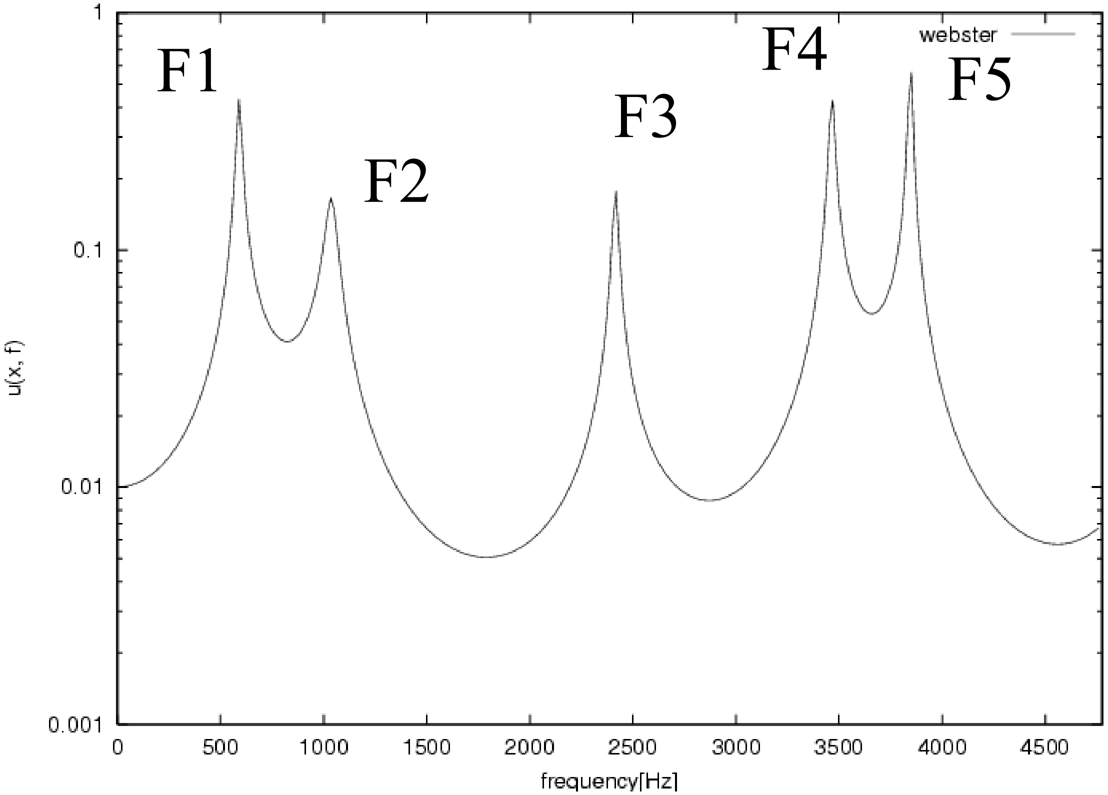
Radiation boundary with Dirichlet to Neumann mapping

★ Numerical example of frequency response function and formants in the case of voice generation (DD15 & [4], [8])

Frequency response at observation point x :

$$W(x,\omega) = \sqrt{\Re u(x,\omega)^2 + \Im u(x,\omega)^2}$$

In the case of vowel /a/

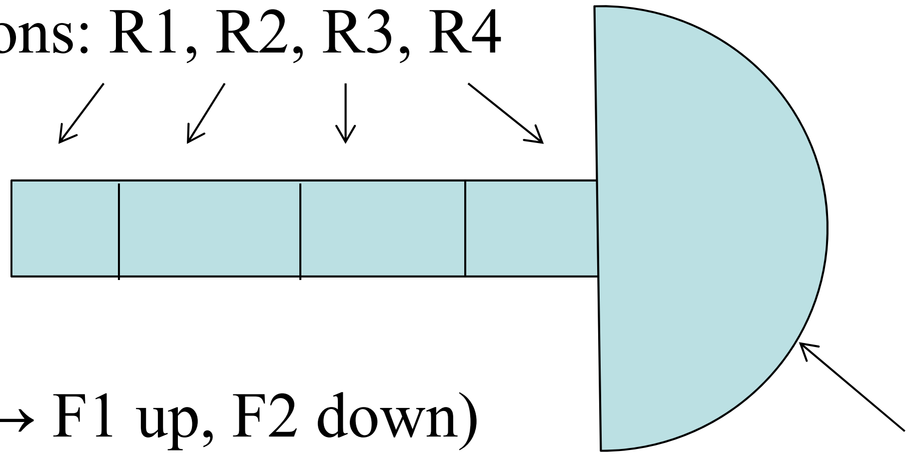


Formant: Peak of frequency response function

Empirically, 3 or 4 lowest formants characterize vowels

● Computation by using FreeFEM++

Bifurcation phenomena from neutral straight waveguide tube with four fundamental regions: R1, R2, R3, R4 (or more)



Neutral : straight tube

Case 1 :region R3 swells (→ F1 up, F2 down)

Case 2 :region R2 swells (→ F1 down, F2 up)

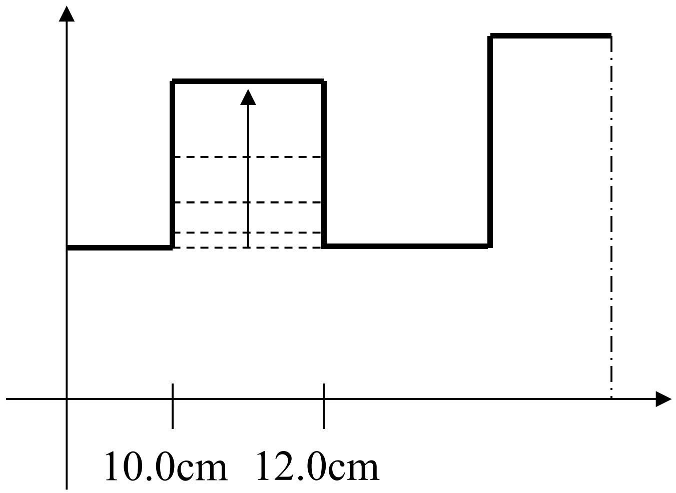
★ FreeFEM++ is an open software having been developed by Paris VI group and others: http://www.freefem.org/ff++/

● Some observations from numerical results:

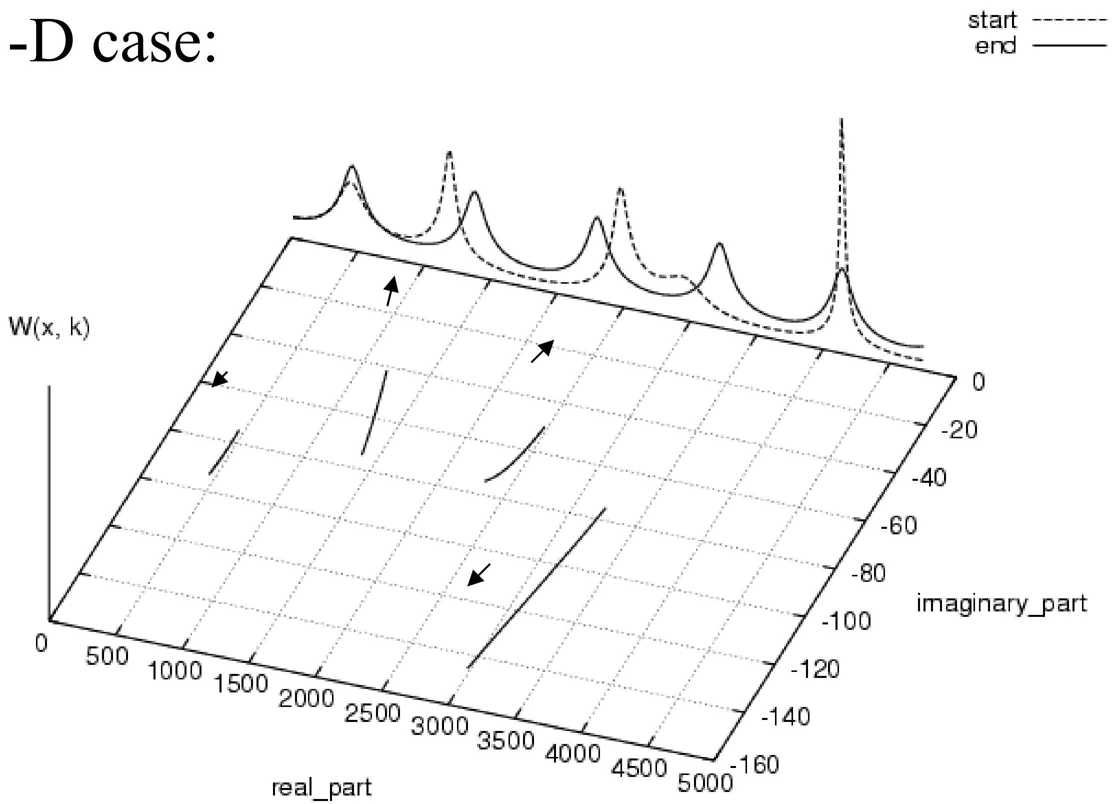
Frequency response function and its peaks are influenced by the corresponding "resonant poles" in the complex plane

Numerical example for 1-D case:

A(x) : area function



Perturbation from neutral shape to a swelled one



Change of frequency response function and **trajectory** of moving **resonant eigenvalues** defined in the next slide

★ Correspondence between formants and complex eigenvalues

Consider the eigenvalue problem for $k = \omega / c$ in C

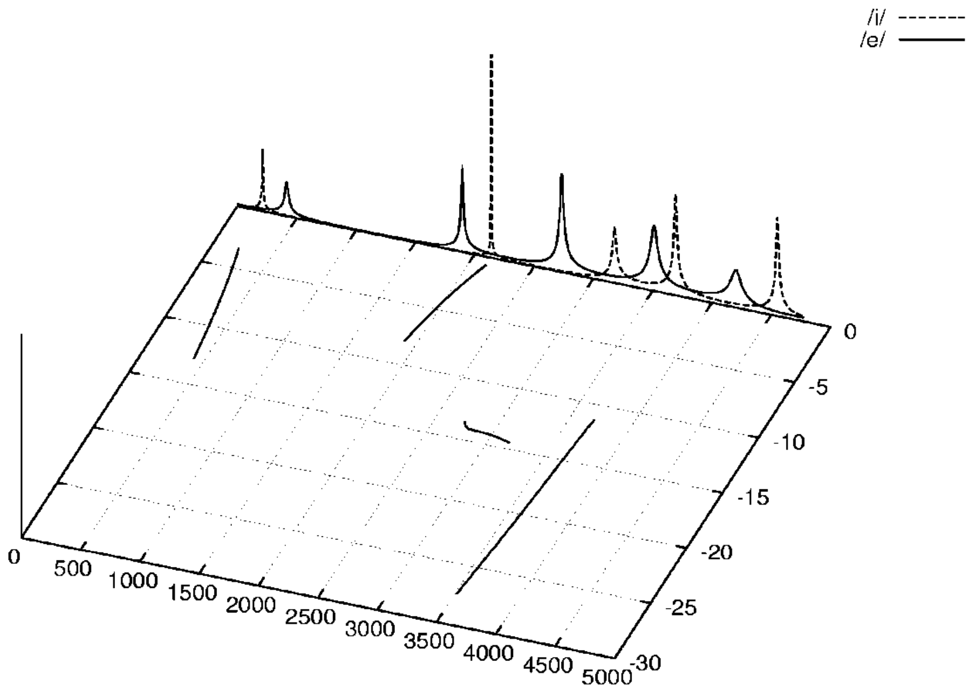
$$-\Delta u = k^2 u \quad in \quad \Omega_R$$

$$\frac{\partial u}{\partial n} = 0 \quad on \quad \partial \Omega$$

$$\frac{\partial u}{\partial r} = -M(k) u \quad on \quad \Gamma_R$$

Real part of k corresponds to the position of Formant, and imaginary part of k to its height or width.

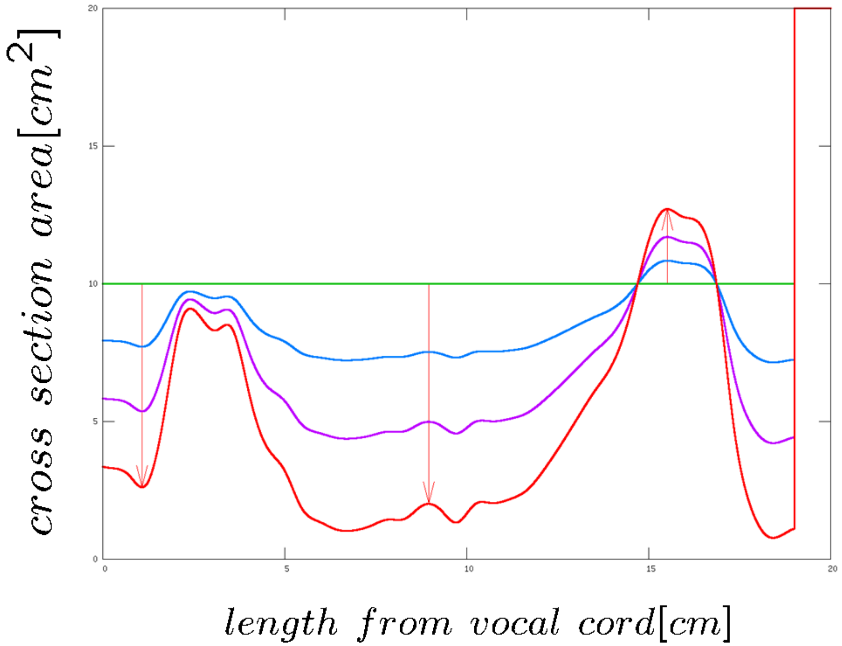
★ Correspondence between complex eigenvalues and frequency response functions

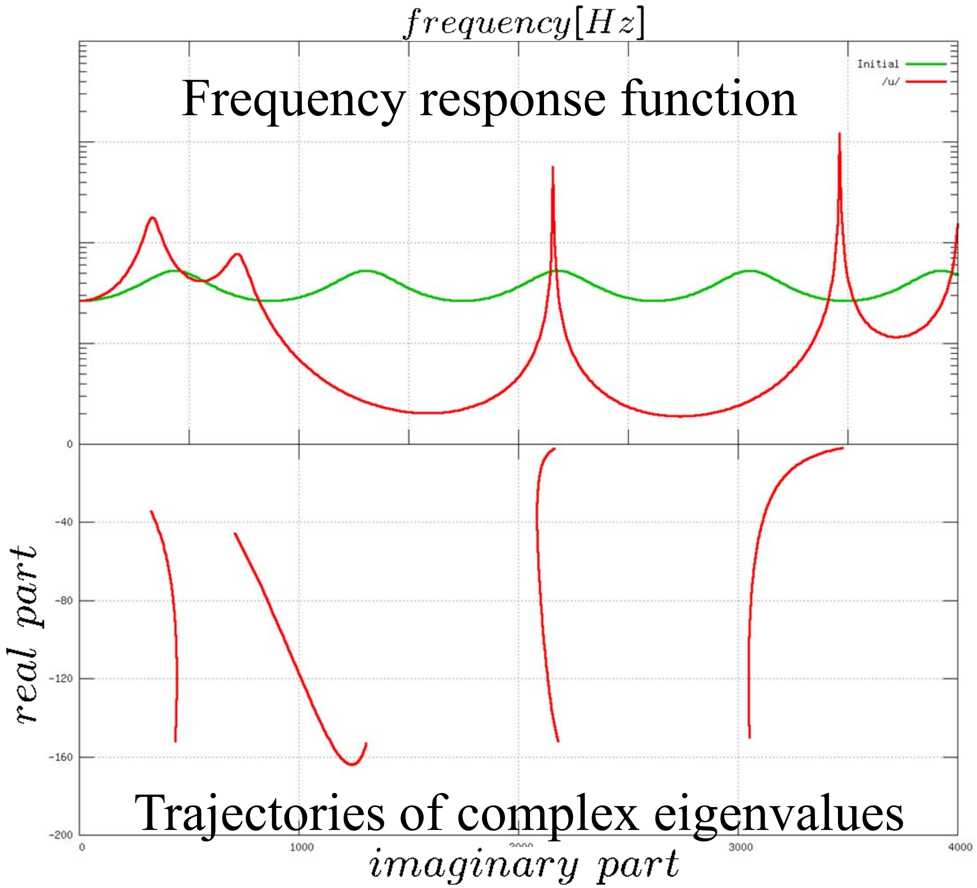


Four lines correspond to four trajectories of complex eigenvalues

Example: Changing of wave guide shape from the neutral shape to another shape and corresponding trajectory of complex eigenvalues and the frequency response function

Changing of wave guide shape





There is a good correspondence between frequency response function and complex eigenvalues

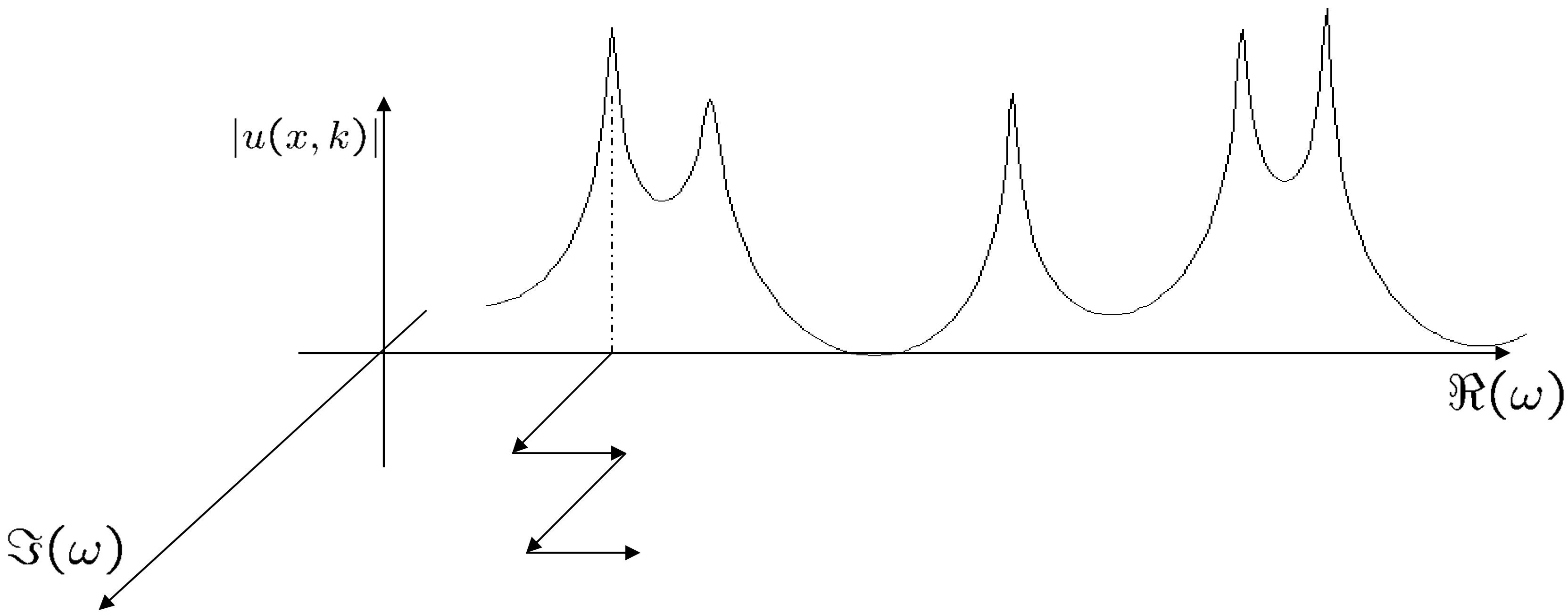
★ Iteration algorithm for computing complex eigenvalues

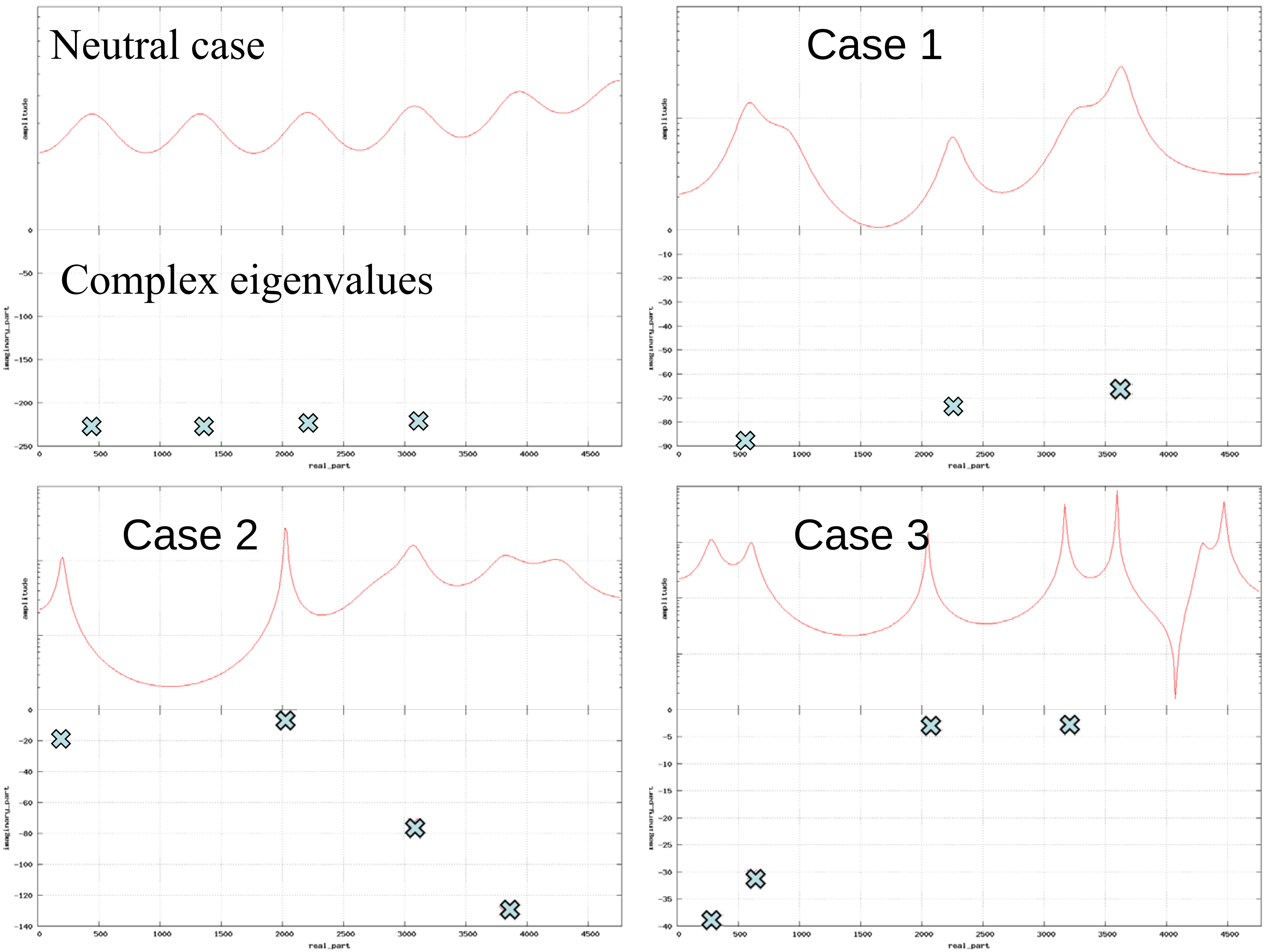
1. Compute frequency response function by FEM
2. Compute N local maximum points (=formants) of frequency response function
3. Search the point that gives the local maximum value of $|u(z)|$ in the complex domain starting from the formants
4. Perform line search through the lines parallel to the real axis and the imaginary axis alternatively
5. Find the pole in the complex domain as the limit point
6. Terminate the procedure when N poles (=complex eigenvalues) are found

★ Example in 2-D case:

$$\begin{cases} -\Delta u - \lambda u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = -\frac{i\omega}{A_0} & \text{on } \Gamma_s \\ \frac{\partial u}{\partial n} = iku & \text{on } \Gamma_r \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma \setminus (\Gamma_s \cup \Gamma_r) \end{cases}$$

$$(\lambda = k^2 = \frac{\omega^2}{c^2})$$




Neutral case
Complex eigenvalues
Case 1
Case 2
Case 3
amplitude
imaginary_part
real_part

★ Resonance eigen-values and inverse problem
related to vocal tract shape and resonance

Theorem(Gårding) Let ω_n (n=1,2,3,…) be resonances of the Webster system: $A(x)p_x + u_t = 0$, $A(x)p_t + u_x = 0$, on [0,1] with boundary conditions $p(0,t) = \delta(1)$ and $A(x)p(1,t) = bu(1,t)$, where $0 < b \equiv A(1)/A_0 (< 1)$, a constant called loss coefficient.

Then, Im ω_n>0, Re $\omega_n \geq 0$ for all n, and there is an asymptotic expansion

$$\omega_n \sim 2^{-1}n - 4^{-1} + ic + c_1 n^{-1} + c_2 n^{-2} + \cdots$$

for large n where $4\pi c = \log((1+b)/(1-b)) > 0$.

Conversely, given such numbers, they are the vowel resonances of a tube with loss coefficient $b = \tan\text{hyp}\ 2\pi$ c and an infinitely differentiable function $A(x)$, unique when a normalized so that $A(1) = 1$.

Reference:

[1] Gårding, L., The inverse of vowel articulation, Ark. Mat., 15.1 (1977), 63-86.
[2] Gel′fand, I.M. and Levitan, M.B., On the determination of a differential equation from its spectral function. AMS, 1955.
[3] Sondhi, M. M., and B. Gopinath, Determination of Vocal ‐ Tract Shape from Impulse Response at the Lips, J. Acoust. Soci. America (1971) 1867-1873.
[3] Kirsch, A., An introduction to the mathematical theory of inverse problems; Chapter 4.5 The inverse problem, Springer, 1996.

★ Sensitivity or perturbation analysis of frequency response with respect to vocal-tract shape variation:

[15] M. R. Schroeder, Determination of the geometry of the human vocal tract by acoustic measurements, The Journal of the Acoustical Society of America, Vol.41, Num.4 (1967) pp.1002-1010.

Ehrenfest's theorem: $\Delta(E_n/f_n) = 0$, where Δ stands for an adiabatic perturbation and the subscript n refers to one of the many linear oscillator modes of the physical system under consideration. For a small perturbation one may write

$$\delta E_n/f_n - E_n \delta f_n/{f_n}^2 = 0, \text{ or } \delta f_n/f_n = \delta E_n/E_n$$

i.e., the relative frequency shift is equal to the relative change in energy of the oscillator. Furthermore, Brillouin has shown that

$$\delta E_n = -\int_0^L P_n(x)\delta A(x)dx, \text{ with } P = p^2/2\rho c^2 - \rho v^2/2$$

P. Ehrenfest, Proc. Amsterdam Acad. 19, 576-597 (1916).
See also Ann. Physik 51, 321-332 (1916); Phil. Mag. 33, 500-513(1917).

★ Perturbation theory and Sensitivity function

Definition of "**sensitivity function**" due to Fant (see[5]) :
Relative frequency shift $\delta F_n / F_n$ of resonance frequencies F_1, F_2, F_3 etc. caused by a perturbation $\delta A(x)/A(x)$ of area function $A(x)$ is referred to as "sensitivity function"

Characterization of "**sensitivity function**" by Fant & Pauli (see[5]):
Sensitivity function for area perturbation of any $A(x)$ is equal to the distribution with respect to x of the difference $E_{kx} - E_{px}$ between the kinetic energy $E_{kx} \equiv \frac{1}{2} L(x) U^2(x)$ and the potential energy $E_{kp} \equiv \frac{1}{2} C(x) P^2(x)$ normalized by the totally stored energy.
Here $U(x)$ is flow, $P(x)$ is pressure and $L(x) \equiv \varrho / A(x)$ is an acoustic inductance and $C(x)$ is some parameter function.

[5] Fant, G., The relations between area functions and the acoustic signal, Phonetica, 37 (1980) pp.55-86.

Variational formula of complex eigenvalues for 1D case

Perturb the area function as

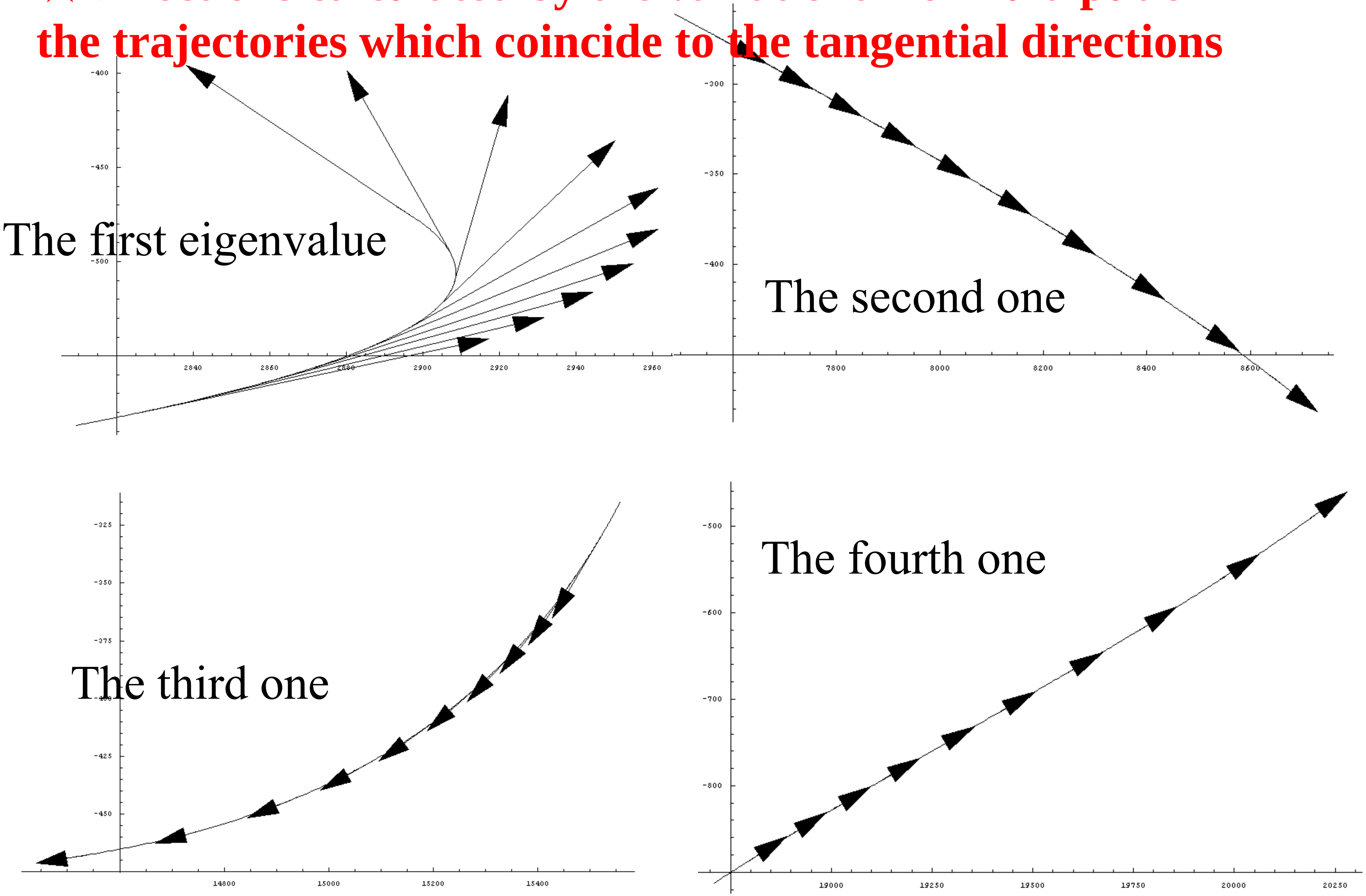
$$\begin{cases} -\frac{d}{dx}((A+\delta A)\frac{d}{dx}(u+\delta u)) \\ \qquad = (k+\delta k)^2(A+\delta A)(u+\delta u) & \text{in } \Omega \\ \frac{d}{dx}(u+\delta u) = 0 & \text{on } \Gamma_s \\ \frac{d}{dx}(u+\delta u) = i(k+\delta k)(u+\delta u) & \text{on } \Gamma_r \end{cases}$$

Modifying the above formula, we can derive the variational formula for $\omega = ck$ and hence $\delta\omega = c\delta k$:

$$\delta\omega(\delta A) = \frac{c(\int_0^L \delta A((\frac{du}{dx})^2 - \frac{\omega^2}{c^2}u^2)dx - i\frac{\omega}{c}\delta A(L)u(L)^2)}{2\frac{\omega}{c}\int_0^L Au^2dx + iA(L)u(L)^2}$$

[10] Kako, T. and Touda, K., Numerical method for voice generation problem based on finite element method, Journal of Computational Acoustics, Vol. 14, No. 1 (2006) 45–56

★Directions calculated by the variational formula put on the trajectories which coincide to the tangential directions



● Vocal tract shape design algorithm

★**Strategy**: to get a vocal tract shape for a given frequency response function by designing the corresponding complex eigenvalues

Vocal tract shape $A_0(\alpha)$: Initially given → $\hat{A}$: Unknown

Complex eigenvalues $\omega_n^{(0)}(\alpha)$: Initially given → ω_n^* : Known target

★ **We design the vocal tract shape matching resonant eigenvalues**:

$$A(\alpha) = A(x:\alpha) = \sum_{k=0}^{M-1} \alpha_k \phi_k(x) \qquad \alpha = \{\alpha_k\}(k = 0, \cdots, M-1)$$

ϕ_k: Basic shape functions,

α_k: design parameters, M: number of parameters

$$Minimize : F(\alpha) = \sum_{n=1}^{N} |\omega_n^* - \omega_n(\alpha)|^2 \qquad \alpha \in R^M$$

N:number of target eigenvalues

Then, we have the expression of variation of area function as

$$\delta A = \sum_{k=0}^{M-1} \delta\alpha_k \phi_k(x)$$

★ Optimization problem

Minimize: $F(\alpha)$

↓

To solve unconstrained optimization problem

★ Conjugate gradient method with the line search, esp. **Polak-Ribiere** method: only gradient is used (see [17])

$$\alpha^{k+1} = \alpha^k + c d^k$$

(c can be determined by line search.)

$$d^k = -\nabla F(\alpha^k) + \frac{(\nabla F(\alpha^k) - \nabla F(\alpha^{k-1}))^t \nabla F(\alpha^k)}{\| \nabla F(\alpha^{k-1}) \|^2} d^{k-1}$$

$$d^0 = -\nabla F(\alpha^0)$$

★ Algorithm to compute gradient ∇F

$$\omega_n(\alpha+\epsilon e_i) \approx \omega_n(\alpha)+\epsilon\delta\omega_n(\phi_i) \quad (0 \le i \le M-1, 1 \le n \le N)$$

$$\omega, \delta\omega \in C^N, \alpha \in R^M, \epsilon \in R$$

$$(\nabla F(\alpha))_i = \lim_{\epsilon\to 0} \frac{F(\alpha+\epsilon e_i) - F(\alpha)}{\epsilon} \qquad (i = 0, \cdots, M-1)$$

$$= \lim_{\epsilon\to 0} \sum_{n=1}^{N} \frac{|\omega_n^* - \omega_n(\alpha+\epsilon e_i)|^2 - |\omega_n^* - \omega_n(\alpha)|^2}{\epsilon}$$

$$= \lim_{\epsilon\to 0} \sum_{n=1}^{N} \frac{(\omega_n^* - \omega_n(\alpha) - \epsilon\delta\omega_n(\phi_i))\overline{(\omega_n^* - \omega_n(\alpha) - \epsilon\delta\omega_n(\phi_i))}}{\epsilon}$$

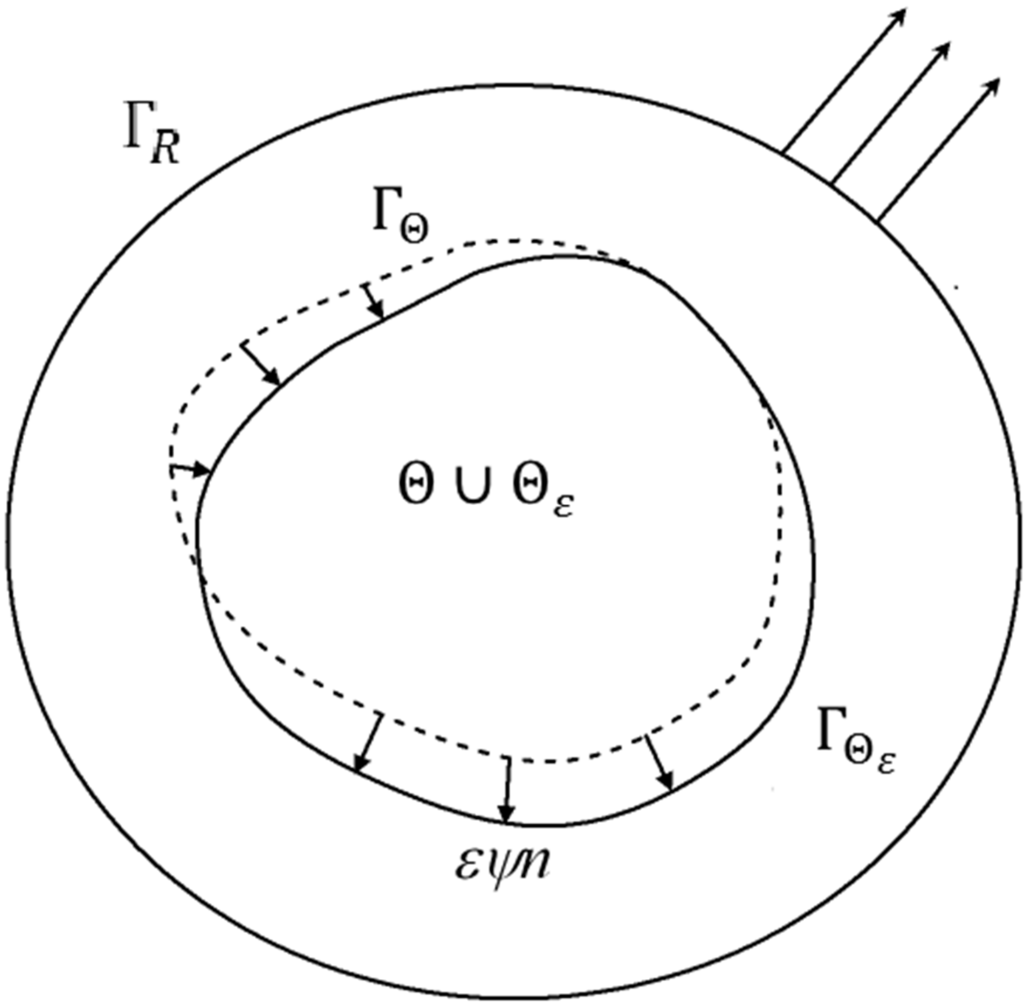
$$- \frac{(\omega_n^* - \omega_n(\alpha))\overline{(\omega_n^* - \omega_n(\alpha))}}{\epsilon}$$

$$= \lim_{\epsilon\to 0} \sum_{n=1}^{N} \frac{-2\Re[(\omega_n^* - \omega_n(\alpha))\overline{\epsilon\delta\omega_n(\phi_i)}] + \epsilon^2|\delta\omega_n(\phi_i)|^2}{\epsilon}$$

$$= \sum_{n=1}^{N} -2\Re[(\omega_n^* - \omega_n(\alpha))\overline{\delta\omega_n(\phi_i)}]$$

Then we can use the variational formula of $\delta\omega_n$ for computing $\nabla F(\alpha)$.

●Variational formula of resonance eigenvalues for 2 and 3 dimensional cases



In the case of bounded domain and problem is self-adjoint and hence the eigenvalues are all real, Hadamard has gotten:

Theorem (Hadamard): The first variation of the Neumann eigenvalues of the Laplacian under domain perturbation is given by

$$\lambda'(0) = \int_{\partial\Omega} (|\nabla_{\partial\Omega} u|^2 - \lambda u^2)\Psi \, dA .$$

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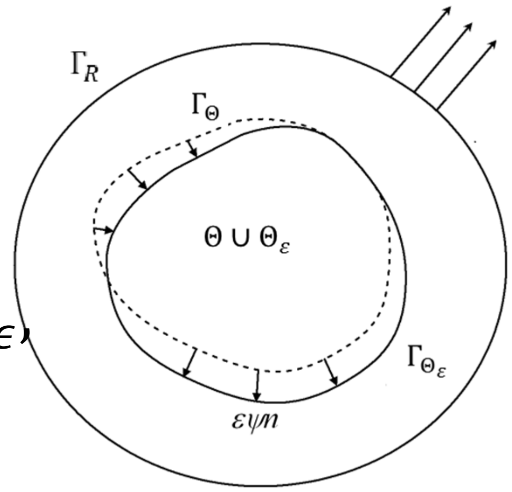


Proof: We start up with the following two equations:

$$-\Delta u_\varepsilon - k(\varepsilon)^2 u_\varepsilon = 0 \ \ in\ \Omega_\varepsilon,\ \ \varepsilon \in [0, \varepsilon_0), \qquad (1)$$

$$-\Delta u \ - \ k^2 u \ = 0 \quad in\ \Omega, \quad u = u_0, k = k(0), \ (2)$$

and taking the difference of these equations in $\Omega \cap \Omega_\epsilon$, we have

$$-\Delta(u - u_\varepsilon) - (k^2 u - k(\varepsilon)^2 u_\varepsilon) = 0 \ \ in\ \Omega \cap \Omega_\varepsilon. \quad (3)$$

Multiplying this equality by u, integrating it over $\Omega \cap \Omega_\varepsilon$, making use of twice integration of parts and the equality $-\Delta u = k^2 u$, we have

$$0 = \int_{\Omega\cap\Omega_\varepsilon} \left(k(\varepsilon)^2 u_\varepsilon - k^2 u\right) u\, dx \ - \int_{\Omega\cap\Omega_\varepsilon} \nabla(u_\varepsilon - u) \cdot \nabla u\, dx$$

$$+ \int_{\partial(\Omega\cap\Omega_\varepsilon)} \{\frac{\partial}{\partial n}(u_\varepsilon - u)\} u\, d\sigma$$

$$= \int_{\Omega\cap\Omega_\varepsilon} \left(k(\varepsilon)^2 u_\varepsilon - k^2 u\right) u\, dx + \int_{\Omega\cap\Omega_\varepsilon} (u_\varepsilon - u) \Delta u\, dx$$

$$+ \int_{\partial(\Omega\cap\Omega_\varepsilon)} \{\frac{\partial}{\partial n}(u_\varepsilon - u)\} u\, d\sigma - \int_{\partial(\Omega\cap\Omega_\varepsilon)} (u_\varepsilon - u) \frac{\partial}{\partial n} u\, d\sigma$$

$$= \int_{\Omega\cap\Omega_\varepsilon} \left(k(\varepsilon)^2 - k^2\right) u u_\varepsilon\, dx + \int_{\partial(\Omega\cap\Omega_\varepsilon)} \left(\frac{\partial u_\varepsilon}{\partial n} u - u_\varepsilon \frac{\partial u}{\partial n}\right) d\sigma \quad (4)$$

We derive the expression of $\frac{dk}{d\varepsilon}(0)$ as follows:

First of all, since we set $k(0) = k$, we have

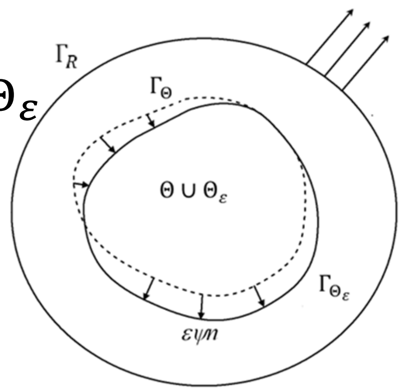
$$k(\varepsilon)^2 - k^2 = (k + \frac{dk}{d\varepsilon}(0)\varepsilon + o(\varepsilon^2))^2 - k^2 = 2k\frac{dk}{d\varepsilon}(0)\varepsilon + o(\varepsilon^2) \quad (5)$$

Using the notation $\Theta \equiv \Omega^c$, $\Theta_\varepsilon \equiv {\Omega_\varepsilon}^c$, we have

$\partial(\Omega \cap \Omega_\varepsilon) = \Gamma_R \cup \partial(\Theta \cup \Theta_\varepsilon)$ and $\partial(\Theta \cup \Theta_\varepsilon) = \Gamma_\Theta \cup \Gamma_{\Theta_\varepsilon}$

with $\Gamma_\Theta \equiv \partial(\Theta \cup \Theta_\varepsilon) \cap \partial\Theta$ and $\Gamma_{\Theta_\varepsilon} \equiv \partial(\Theta \cup \Theta_\varepsilon) \cap \partial\Theta_\varepsilon$

and radiation boundary Γ_R.



Now we have for the second term of (4)

$$\int_{\partial(\Theta\cup\Theta_\varepsilon)} \left(\frac{\partial u_\varepsilon}{\partial n} u - u_\varepsilon \frac{\partial u}{\partial n}\right) d\sigma = \int_{\Gamma_\Theta \backslash \partial\Theta_\varepsilon} \frac{\partial u_\varepsilon}{\partial n} u \, d\sigma - \int_{\Gamma_{\Theta_\varepsilon} \backslash \partial\Theta} u_\varepsilon \frac{\partial u}{\partial n} \, d\sigma, \quad (6)$$

as $\frac{\partial u}{\partial n} = 0$ on Γ_Θ and $\frac{\partial u_\varepsilon}{\partial n} = 0$ on $\Gamma_{\Theta_\varepsilon}$ by respective homogeneous Neumann boundary condition.

Remark: In the case of homogeneous Dirichlet condition, we have

$$\int_{\partial(\Theta\cup\Theta_\varepsilon)} \left(\frac{\partial u_\varepsilon}{\partial n} u - u_\varepsilon \frac{\partial u}{\partial n}\right) d\sigma = -\int_{\Gamma_\Theta \backslash \partial\Theta_\varepsilon} u_\varepsilon \frac{\partial u}{\partial n} d\sigma + \int_{\Gamma_{\Theta_\varepsilon} \backslash \partial\Theta} \frac{\partial u_\varepsilon}{\partial n} u \, d\sigma. (7)$$

Furthermore, since

$n(x+\varepsilon\Psi(x)n(x))\}\cdot\ \nabla u_\varepsilon\big(x+\varepsilon\Psi(x)n(x)\big)=0$ on $\Gamma_{\Theta_\varepsilon}$,

we can estimate the first term of the last expression in (6) as follows:

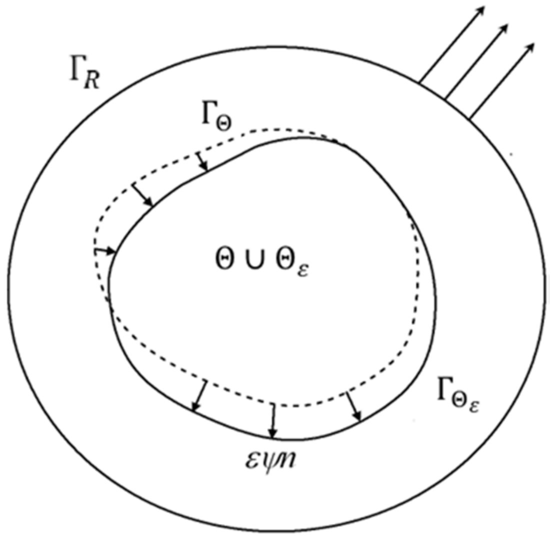
$$\int_{\Gamma_\Theta\backslash\partial\Theta_\varepsilon}\frac{\partial u_\varepsilon}{\partial n}u\,d\sigma=\int_{\partial(\Theta\cap\Omega_\varepsilon)}\frac{\partial u_\varepsilon}{\partial n}u\,d\sigma=\int_{\Theta\cap\Omega_\varepsilon}(\Delta u_\varepsilon)u+\nabla u_\varepsilon\cdot\nabla u\,dx$$

$$=\int_{\Theta\cap\Omega_\varepsilon}-k(\varepsilon)^2u_\varepsilon u+\nabla u_\varepsilon\cdot\nabla u\,dx=\int_{\Theta\cap\Omega_\varepsilon}-k(\varepsilon)^2u_\varepsilon u+\nabla u_\varepsilon\cdot\nabla u\,dx.$$

Then, as ε tends to zero, we have

$$\int_{\Theta\cap\Omega_\varepsilon}-k(\varepsilon)^2u_\varepsilon u+\nabla u_\varepsilon\cdot\nabla u\,dx=$$

$$\varepsilon\int_{\Gamma_{\Theta\cap\{\sigma\,|\Psi(\sigma)<0\}}}\{\nabla u\cdot\nabla u-k^2u^2\}\Psi(\sigma)d\sigma+o(\varepsilon^2).$$



Similarly, since $n(x)\cdot\nabla u(x)=0$ on Γ_Θ, we have

$$-\int_{\Gamma_\Theta\backslash\partial\Theta_\varepsilon}u_\varepsilon\frac{\partial u}{\partial n}d\sigma=\varepsilon\int_{\Gamma_{\Theta\cap\{\sigma\,|\Psi(\sigma)\geq 0\}}}\{\nabla u\cdot\nabla u-k^2u^2\}\Psi(\sigma)d\sigma+o(\varepsilon^2).$$

Consequently, we have

$$0 = \int_{\Omega\cap\Omega_\varepsilon}\left(k(\varepsilon)^2 - k^2\right)uu_\varepsilon\, dx + \int_{\partial(\Omega\cap\Omega_\varepsilon)}\left(\frac{\partial u_\varepsilon}{\partial n}u - u_\varepsilon\frac{\partial u}{\partial n}\right)d\sigma$$

$$=2k\varepsilon\frac{dk}{d\varepsilon}(0)\int_\Omega u^2\, dx + \varepsilon\int_{\partial\Theta}\{\nabla u\cdot\nabla u - k^2u^2\}\Psi(\sigma)d\sigma$$

$$+\int_{\partial\Gamma_R}\left(\frac{\partial u_\varepsilon}{\partial n}u - u_\varepsilon\frac{\partial u}{\partial n}\right)d\sigma + o(\varepsilon^2).$$

Using Dirichlet to Neumann mapping on Γ_R and its derivative w.r.t. k, we have

$$\int_{\partial\Gamma_R}\left(\frac{\partial u_\varepsilon}{\partial n}u - u_\varepsilon\frac{\partial u}{\partial n}\right)d\sigma = \int_{\partial\Gamma_R}\{(M(k(\varepsilon))u_\varepsilon)u - u_\varepsilon M(k)u\}d\sigma$$

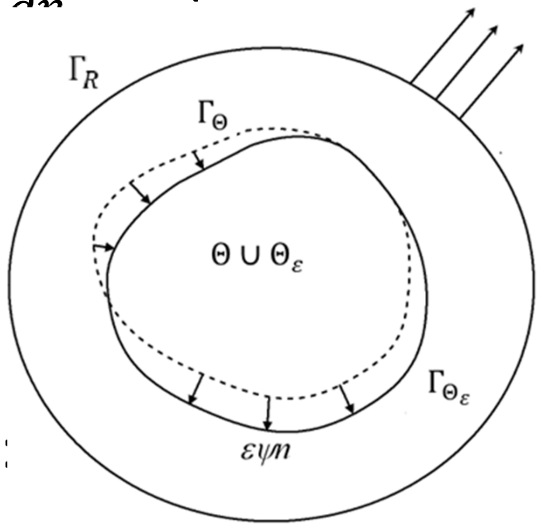
$$= \varepsilon\frac{dk}{d\varepsilon}(0)\int_{\partial\Gamma_R}\left\{\frac{\delta M}{\delta k}(k)u\right\}u\,d\sigma + O(\varepsilon^2).$$

Here, we have used the complex symmetric property of DtN mapping.

Combining these results and noting the fact $\frac{\partial}{\partial n}u$ =(n· $\nabla)u = 0$ on $\partial\Theta$ and hence

$$\nabla u = \nabla u - (n\cdot\nabla)u \equiv \nabla_{\perp}u \text{ on } \partial\Theta,$$

we finally obtain the result of variational formula of resonance eigenvalue:



$$\frac{dk}{d\varepsilon}(0) = \frac{\int_{\partial\Theta}\{|\nabla_{\perp}u|^2 - k^2u^2\}\Psi(\sigma)d\sigma}{2k\int_{\Omega} u^2\,dx + \int_{\partial\Gamma_R}\left\{\frac{\delta M}{\delta k}(k)u\right\}ud\sigma}.$$

Conclusion

- We reviewed some numerical methods for wave guide problem using finite element method based on the Helmholtz equation for time harmonic wave propagation.

- We confirmed the relation between the frequency response function and the complex eigenvalues.

- We introduced the variational formula for resonance eigenvalues with respect to a small perturbation of boundary, and confirmed the validity of numerical method for this formula.

- We considered the optimization problem to coincide with the complex eigenvalue, and we proposed an algorithm to design the wave guide shape based on this optimization problem using the above formulation.

Thank you for your attention!

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