# Immersed finite element method for eigenvalue problems in elasticity


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## 1 Eigenvalue problems in elasticity

Eigenvalue analysis is essential basis for many types of engineering analysis. As eigenvalues are closely related with the frequency and shape of structures, computing the eigensolutions is important to interpret the dynamic interaction between the structures. If the frequency of structures is close to the system's natural frequency, mechanical resonance occurs. It may lead to catastrophic failure or damage in constructed structures such as bridges, buildings, and towers.

The model problem is described as :

- $\Omega$ is a connected and convex polygonal domain.
- $\Omega^+$ and $\Omega^-$ are subdomains in $\Omega$ and divided by a interface $\Gamma = \partial\Omega^+ \cap \partial\Omega^-$.
- $\lambda$ and $\mu$ denote the Lamé coefficients ($0 < \mu_1 < \mu < \mu_2$ and $0 < \lambda < \infty$).
- The Cauchy stress tensor $\boldsymbol{\sigma} := (\sigma_{ij})$ and linearized strain tensor $\boldsymbol{\epsilon} := (\epsilon_{ij})$ in $\mathbb{R}^{2\times 2}$ are given by

$$\boldsymbol{\sigma}(\mathbf{u}) = 2\mu\,\boldsymbol{\epsilon}(\mathbf{u}) + \lambda\, tr(\boldsymbol{\epsilon}(\mathbf{u}))\boldsymbol{I}, \quad \boldsymbol{\epsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T).$$

The eigenvalue problem for the linear elasticity equation with interface is

$$-\text{div}\,\boldsymbol{\sigma}(\mathbf{u}) = \omega^2\mathbf{u} \quad \text{in } \Omega^s \quad (s = +, -), \tag{1.1}$$
$$[\mathbf{u}]_\Gamma = 0, \tag{1.2}$$
$$[\boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n}]_\Gamma = 0, \tag{1.3}$$
$$\mathbf{u} = 0 \quad \text{on } \partial\Omega,$$

where $\omega^2$ and $\mathbf{u}$ are the corresponding eigenvalue and eigenfunction, and the symbol $[\cdot]$ denotes the jump across the interface $\Gamma$.

We formulate the model problem (1.1) into the displacement formulation

$$a(\mathbf{u},\mathbf{v}) = \omega^2(\mathbf{u},\mathbf{v}), \tag{1.4}$$

where

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega 2\mu\,\boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v})dx + \int_\Omega \lambda\,\text{div}\,\mathbf{u}\,\text{div}\,\mathbf{v}\,dx, \quad \omega^2(\mathbf{u},\mathbf{v}) = \omega^2\int_\Omega \mathbf{u}\cdot\mathbf{v}dx.$$

## 2 Immersed finite element method (IFEM)

We introduce an immersed finite element method based on Crouzeix-Raviart elements.

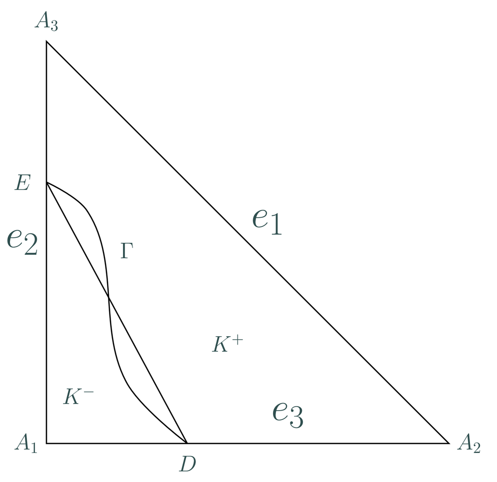


**Figure 1:** A typical interface triangle

$$\frac{1}{|e_j|}\int_{e_j}\hat{\phi}_{i1}\,ds = \delta_{ij}, \qquad j = 1,2,3,$$
$$\frac{1}{|e_j|}\int_{e_j}\hat{\phi}_{i2}\,ds = \delta_{(i-3)j}, \quad j = 1,2,3,$$
$$[\widehat{\boldsymbol{\phi}}_i(D)] = 0,$$
$$[\widehat{\boldsymbol{\phi}}_i(E)] = 0,$$
$$\Big[\boldsymbol{\sigma}(\widehat{\boldsymbol{\phi}}_i)\cdot\mathbf{n}\Big]_{\overline{DE}} = 0.$$

For an interface element $K$ (see Figure 1), the piecewise linear basis function $\widehat{\boldsymbol{\phi}}_i = (\hat{\phi}_{i1}, \hat{\phi}_{i2})$, $i = 1, 2, \cdots, 6$, satisfies the interface conditions (1.2), (1.3).

- $\widehat{\mathbf{N}}_h(\Omega)$ : IFEM space spanned by basis $\widehat{\boldsymbol{\phi}}$.
- $\mathbf{H}_h(\Omega) := (H_0^1(\Omega))^2 + \widehat{\mathbf{N}}_h(\Omega)$.

The IFEM for the eigenvalue problem (1.1) is to find the eigensolution $(\omega_h^2, \mathbf{u}_h) \in \mathbb{C}\times\widehat{\mathbf{N}}_h(\Omega)$ such that

$$a_h(\mathbf{u}_h,\mathbf{v}_h) = \omega_h^2(\mathbf{u}_h,\mathbf{v}_h), \qquad \forall\mathbf{v}_h \in \widehat{\mathbf{N}}_h(\Omega), \tag{2.1}$$

where

$$a_h(\mathbf{u},\mathbf{v}) := \sum_{K\in\mathcal{K}_h}\int_K 2\mu\,\boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v})dx + \sum_{K\in\mathcal{K}_h}\int_K \lambda\,\text{div}\,\mathbf{u}\,\text{div}\,\mathbf{v}\,dx + \sum_{e\in\mathcal{E}_h}\frac{\tau}{h}\int_e[\mathbf{u}][\mathbf{v}]ds, \quad \forall\mathbf{u},\mathbf{v}\in\mathbf{H}_h(\Omega).$$

## 3 Spectral approximation

We introduce the solution operator $T : (L^2(\Omega))^2 \to (H_0^1(\Omega))^2$, which associates the solution $T\mathbf{f} \in (H_0^1(\Omega))^2$ of the following source problem with every $\mathbf{f} \in (L^2(\Omega))^2$:

$$a(T\mathbf{f},\mathbf{v}) = (\mathbf{f},\mathbf{v}), \quad \forall\mathbf{v}\in(H_0^1(\Omega))^2.$$

- The operator $T$ : bounded, self-adjoint and compact.
- $(\omega^2,\mathbf{u}) \in \mathbb{C}\setminus\{0\}\times(H_0^1(\Omega))^2$ is an eigenpair of (1.4), $\Leftrightarrow$ $(1/\omega^2,\mathbf{u})$ is an eigenpair of $T$.

In a similar way, the corresponding discrete solution operator $T_h : (L^2(\Omega))^2 \to \widehat{\mathbf{N}}_h(\Omega)$ is defined by

$$a_h(T_h\mathbf{f},\mathbf{v}_h) = (\mathbf{f},\mathbf{v}_h), \quad \forall\mathbf{v}_h\in\widehat{\mathbf{N}}_h(\Omega)$$

with $\mathbf{f} \in (L^2(\Omega))^2$. Clearly,

- The operator $T_h$ : bounded, self-adjoint, and compact.
- $\omega_h^2$ is an eigenvalue from (2.1) $\Leftrightarrow$ $\xi_h = 1/\omega_h^2$ is an eigenvalue of $T_h$.

We prove the spectrally correct approximation of the IFEM by the spectral properties of compact and self-adjoint operators in Banach space.

**Theorem 3.1.** • *Non-pollution of the spectrum*
- *Non-pollution of the eigenspace*
- *Completeness of the eigenspace*
- *Completeness of the spectrum*

**Theorem 3.2.** *Let $\xi$ be an eigenvalue of $T$ with multiplicity $n$. Then for $h$ small enough there exist $n$ eigenvalues $\{\xi_{1,h},\ldots,\xi_{n,h}\}$ of $T_h$ which converge to $\xi$ as follows*

$$\sup_{1\le i\le n}|\xi - \xi_{i,h}| \le Ch^2,$$

*where a positive constant $C$ is independent of $\xi$ and $h$.*

## 4 Numerical results

### Straight-line interface

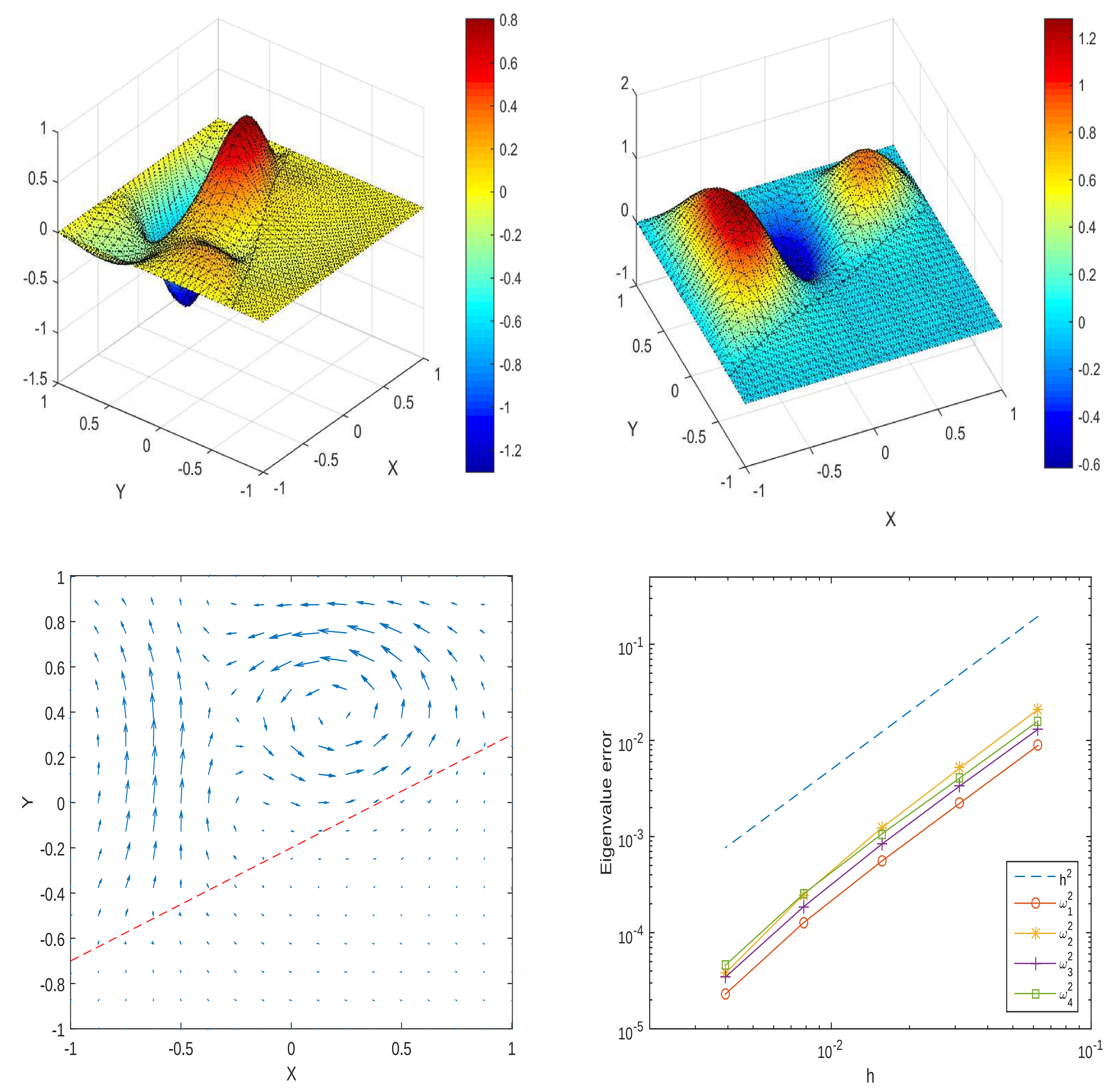


**Figure 2:** Eigenfunction of $\omega_4^2$ when $(\mu^-,\mu^+) = (0.5, 5)$, $\lambda^\pm = 5\mu^\pm$ with straight-line interface. x,y-component of eigenfunction (above). The log-log plots of $h$ versus the relative error of the first four eigenvalues (below on the right).

### Multiple interfaces

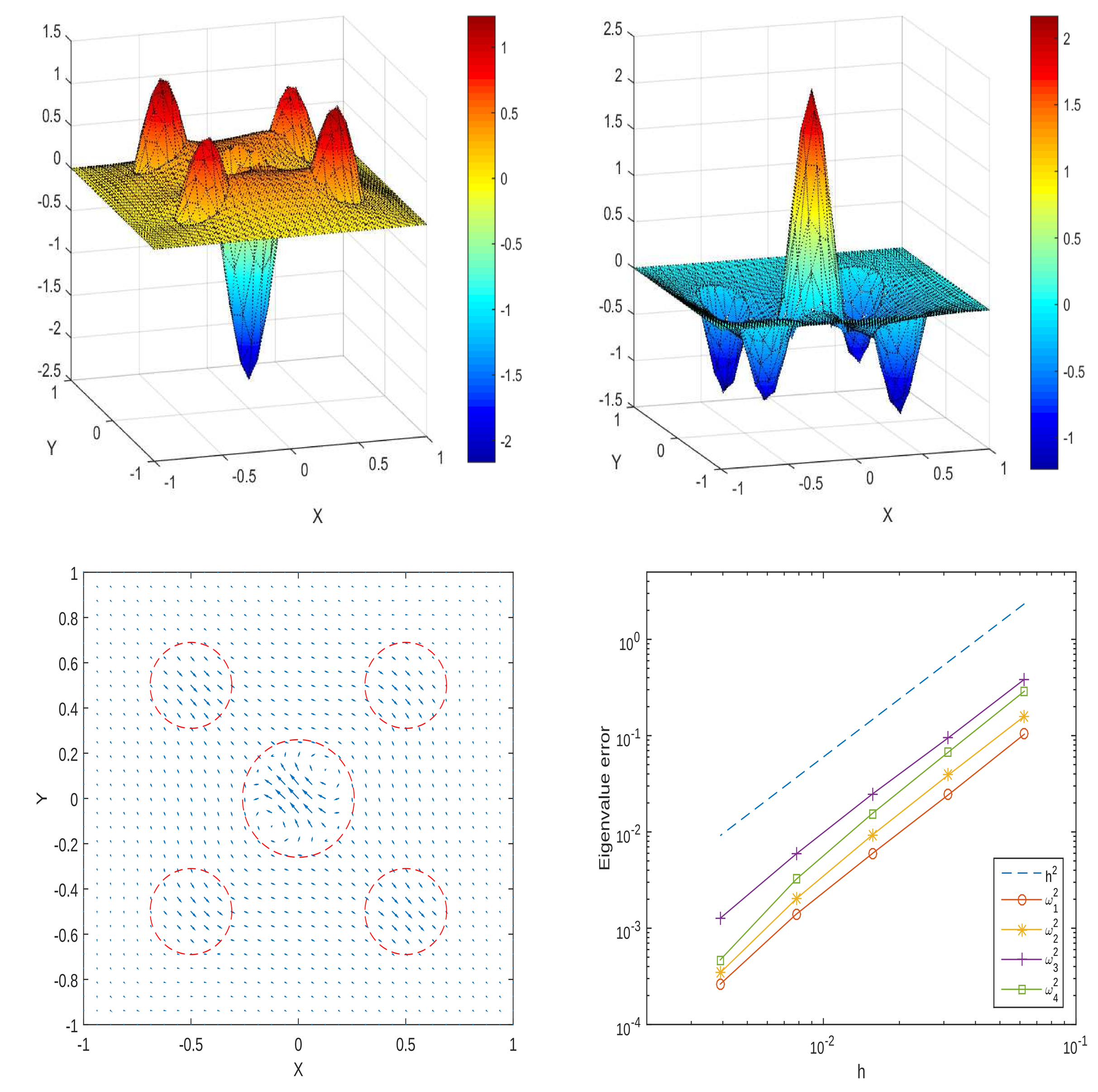


**Figure 3:** Eigenfunction of $\omega_4^2$ when $(\mu^-,\mu^+) = (0.5, 5)$, $\lambda^\pm = 5\mu^\pm$ with multiple interfaces. x,y-component of eigenfunction (above). The log-log plots of $h$ versus the relative error of the first four eigenvalues (below on the right).

## References


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