A Domain Decomposition Method for Quasilinear Elliptic PDEs Using Mortar Finite Elements

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Der Wissenschaftsfonds.

IGDK 1754

Outline

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Motivation

See [Berningner, 2008].

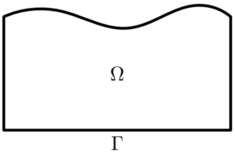
To describe the flow of fluid (water) in porous media, we use the Richards equation for the *physical pressure* p.

Richards Equation

$$n(\mathbf{x})\frac{\partial\theta(p(\mathbf{x},t))}{\partial t} - \nabla\cdot\left(\frac{K(\mathbf{x})}{\mu}k_r\big(\theta(p(\mathbf{x},t))\big)\nabla\big(p(\mathbf{x},t)-\varrho g\, z\big)\right) = f(\mathbf{x},t)$$

Quantities

- θ . . . *saturation*
- ϱ . . . *density*
- K . . . *permeability*
- g . . . *gravitational const.*
- n . . . *porosity*
- μ . . . *viscosity*
- k_r . . . *relative permeability*
- f . . . *source term*



Motivation

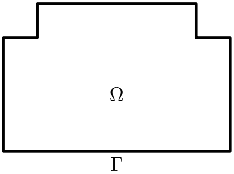
Consider the heat equation to describe the distribution of heat where the thermal conductivity depends on the *heat* p.

Heat Equation

$$\frac{\partial p(\mathbf{x},t)}{\partial t} - \nabla \cdot \Big(k\big(p(\mathbf{x},t)\big) \nabla p(\mathbf{x},t) \Big) = f(\mathbf{x},t)$$

Quantities

k . . . thermal conductivity *f . . . heat source*



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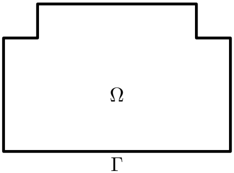

Both equtions have same second order term $-\nabla \cdot \Big(k\left(p(\mathbf{x},t)\right)\,\nabla p(\mathbf{x},t)\Big)$.

Motivation

Let $\Omega \subset \mathbb{R}^d$. For given data f, g_N, g_D consider the following quasilinear boundary value problem.

Model Problem

Find p, such that

$$\begin{aligned} -\nabla \cdot \Big(k\big(p(\mathbf{x})\big) \, \nabla p(\mathbf{x}) \Big) &= f(\mathbf{x}) && \text{in } \Omega \\ p(\mathbf{x}) &= g_D(\mathbf{x}) && \text{on } \Gamma_D \\ k\big(p(\mathbf{x})\big) \, \nabla p(\mathbf{x}) \cdot \mathbf{n}_{\Gamma_N} &= g_N(\mathbf{x}) && \text{on } \Gamma_N \end{aligned}$$

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$$p(\mathbf{x}) = g_D(\mathbf{x}) \qquad \text{on } \Gamma_D$$

$$k\big(p(\mathbf{x})\big)\ \nabla p(\mathbf{x}) \cdot \mathbf{n}_{\Gamma_N} = g_N(\mathbf{x}) \qquad \text{on } \Gamma_N$$

Assumptions on $k : \mathbb{R} \to \mathbb{R}$.

- •) $k \in L_\infty(\mathbb{R})$, i.e. $|k(s)| \leq C < \infty$ for almost all $s \in \mathbb{R}$.
- •) there exists a constant $c > 0$ such that $0 < c \leq k(s)$ for almost all $s \in \mathbb{R}$.
- •) k is Lipschitz continuous, i.e. $|k(s) - k(t)| \leq L\ |s - t|$.

Motivation

Consider the weak formulation of the BVP.

Weak Formulation

Find $p \in H^1_{g_D,\Gamma_D}(\Omega)$, such that

$$\int_\Omega k(p)\, \nabla p \cdot \nabla v \, \mathrm{d}x = \int_\Omega f\, v \, \mathrm{d}x + \int_{\Gamma_N} g_N \, v \, \mathrm{d}s_x$$

is satisfied for all $v \in H^1_{0,\Gamma_D}(\Omega)$.

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is satisfied for all $v \in H^1_{0,\Gamma_D}(\Omega)$.

We can apply the Kirchhoff transformation to the physical quantity p and introduce the *generalized quantity* u as

$$u(\mathbf{x}) := \kappa\big(p(\mathbf{x})\big) = \int_0^{p(\mathbf{x})} k(s)\,\mathrm{d}s.$$

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$$u(\mathbf{x}) := \kappa\big(p(\mathbf{x})\big) = \int_0^{p(\mathbf{x})} k(s)\,\mathrm{d}s.$$

Therefore we get

$$\nabla u(\mathbf{x}) = \kappa'\big(p(\mathbf{x})\big)\,\nabla p(\mathbf{x}) = k\big(p(\mathbf{x})\big)\,\nabla p(\mathbf{x}).$$

Motivation

Transformed Model Problem

Find $u \in H^1_{h_D,\Gamma_D}(\Omega)$, such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f\, v \,\mathrm{d}x + \int_{\Gamma_N} g_N\, v \,\mathrm{d}s_x$$

is satisfied for all $v \in H^1_{0,\Gamma_D}(\Omega)$ with $h_D := \kappa(g_D)$.

- •) Unique solvability from Lax–Milgram Lemma.
- •) Numerical analysis is well known for this problem.
- •) Apply inverse Kirchhoff transformation to obtain the physical quantity p.

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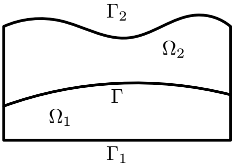
$\Rightarrow$ Try nonlinearities of the form $k : \mathbb{R} \times \Omega \to \mathbb{R}$.

Outline

Approach Continuous Formulation

Consider a second order term is of the form $-\nabla \cdot \Big(k\big(p(\mathbf{x}), \mathbf{x}\big)\ \nabla p(\mathbf{x}) \Big)$.

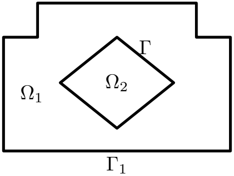
Richards equation



Different soil parameter.

$\Rightarrow$ Different permeabilities.

Heat equation



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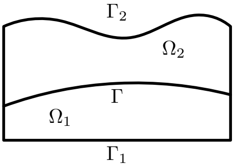

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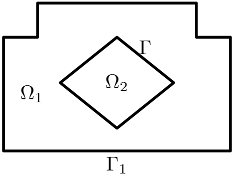

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•) Apply Kirchhoff transformation.

$$\Rightarrow \quad u(\mathbf{x}) := \kappa\big(p(\mathbf{x})\big) = \int_0^{p(\mathbf{x})} k(s, \mathbf{x})\, \mathrm{d}s \quad \textit{but} \quad \nabla u(\mathbf{x}) \neq k\big(p(\mathbf{x}), \mathbf{x}\big)\ \nabla p(\mathbf{x}).$$

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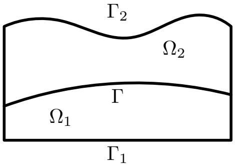

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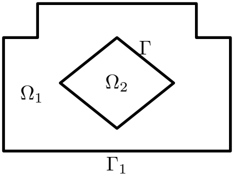

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$\Rightarrow$ new approach to exploit the advantages of the Kirchhoff transformation.

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$$
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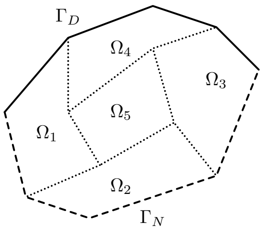

Assumption:

- •) $k\big(p(\mathbf{x}),\mathbf{x}\big) = \sum_{i=1}^{K} \mathcal{X}_{\Omega_i}(\mathbf{x})\, k_i\big(p(\mathbf{x})\big)$
- •) $k_i \in L_\infty(\mathbb{R})$ and $0 < c_i \leq k_i$ for $c_i \in \mathbb{R}$,
- •) k_i is Lipschitz continuous.

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Nonlinear Variational Formulation

Find $p \in H^1_{g_D,\Gamma_D}(\Omega)$, such that

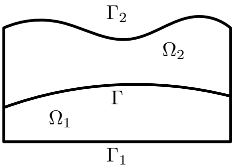
$$\widetilde{a}(p,v) = \big(f,v\big)_{0,\Omega} + \big(g_N,v\big)_{0,\Gamma_N} \qquad \forall\, v \in H^1_{0,\Gamma_D}(\Omega)$$

The linear form $\widetilde{a}(\cdot,\cdot)$ is given by

$$\widetilde{a}(p,v) := \int_\Omega k(p)\nabla p \cdot \nabla v \,\mathrm{d}x.$$

Approach Continuous Formulation

Use Primal Hybrid Formulation to exploit the structure of the nonlinearity.
See [Raviart, 1977].

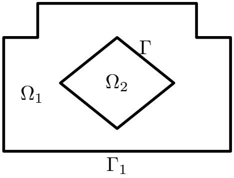


Introduce

$$X := \left\{ v \in L_2(\Omega) \mid v_i \in H^1(\Omega_i), i = 1, \ldots, N \right\},$$

and

$$M_{0,\Gamma_N} := \left\{ \mu \in \prod_{i=1}^{N} H^{\frac{1}{2}}(\partial\Omega_i)' \mid \exists\, \mathbf{q} \in H_{0,\Gamma_N}(div, \Omega) : \mathbf{q} \cdot \mathbf{n}_i = \mu \text{ on } \partial\Omega_i,\ i = 1, \ldots, N \right\}.$$



Approach Continuous Formulation

The variational problem can be written in the equivalent formulation.

Primal Hybrid Variational Formulation

Find $(p, \lambda) \in X \times M_{0,\Gamma_N}$, such that

$$\widehat{a}(p, v) + b(v, \lambda) = \left(f, v\right)_{0,\Omega} + \left(g_N, v\right)_{0,\Gamma_N} \qquad \forall\, v \in X$$

$$b(p, \mu) = -\langle \widetilde{g}_D, \mu\rangle_{\partial\Omega} \qquad \forall\, \mu \in M_{0,\Gamma_N}$$

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The linear form $\widehat{a}(\cdot,\cdot)$ and the bilinear form $b(\cdot,\cdot)$ are given by

$$\widehat{a}(p,v) := \sum_{i=1}^{N} a_i(p_i,v_i) = \sum_{i=1}^{N} \int_{\Omega_i} k_i(p_i)\,\nabla p_i \cdot \nabla v_i \,\mathrm{d}x$$

$$b(p,\mu) := \sum_{i=1}^{N} b_i(p,\mu) = -\sum_{i=1}^{N} \langle \gamma^0_{\partial\Omega_i} p_i, \mu\rangle_{\partial\Omega_i}$$

The bilinear form $b(\cdot,\cdot)$ “enforces” the test function v and the solution u to be continuous (in a weak sense) on the interface and therefore to be in $H^1(\Omega)$.

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$$b(p, \mu) = -\,\langle \widetilde{g}_D, \mu \rangle_{\partial\Omega} \qquad \forall\, \mu \in M_{0,\Gamma_N}$$

Apply the Kirchhoff transformation in each subdomain separately.

$$\Rightarrow \quad u_i(\mathbf{x}) := \kappa_i\big(p_i(\mathbf{x})\big) = \int_0^{p_i(\mathbf{x})} k_i(s)\,\mathrm{d}s \quad \text{and} \quad \nabla u_i(\mathbf{x}) = k_i\big(p_i(\mathbf{x})\big)\,\nabla p_i(\mathbf{x})$$

$$\Rightarrow \quad p_i(\mathbf{x}) := \kappa_i^{-1}\big(u_i(\mathbf{x})\big)$$

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$$b(p,\mu) = -\langle \widetilde{g}_D, \mu\rangle_{\partial\Omega} \qquad \forall\, \mu \in M_{0,\Gamma_N}$$

Rewrite the linear form $\widehat{a}(\cdot,\cdot)$ and the bilinear form $b(\cdot,\cdot)$ in terms of u.

$$\widehat{a}(p,v) = \sum_{i=1}^{N} \int_{\Omega_i} k_i(p_i)\,\nabla p_i \cdot \nabla v_i \,\mathrm{d}x = \sum_{i=1}^{N} \int_{\Omega_i} \nabla u_i \cdot \nabla v_i \,\mathrm{d}x =: a(u,v)$$

$$b(p,\mu) = -\sum_{i=1}^{N} \langle \gamma^0_{\partial\Omega_i}\, p_i, \mu\rangle_{\partial\Omega_i} = -\sum_{i=1}^{N} \langle \gamma^0_{\partial\Omega_i}\, \kappa_i^{-1}(u_i), \mu\rangle_{\partial\Omega_i} =: c(u,\mu)$$

Approach Continuous Formulation

We end up with the following variational problem.

Transformed Nonlinear Primal Hybrid Variational Formulation

Find $(u, \lambda) \in X \times M_{0,\Gamma_N}$, such that

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The above variational problem is equivalent to the variational problem

Nonlinear Variational Formulation

Find $p \in H^1_{g_D,\Gamma_D}(\Omega)$, such that

$$\widetilde{a}(p,v) = (f,v)_{0,\Omega} + (g_N, v)_{0,\Gamma_N} \qquad \forall\, v \in H^1_{0,\Gamma_D}(\Omega)$$

which is uniquely solvable.

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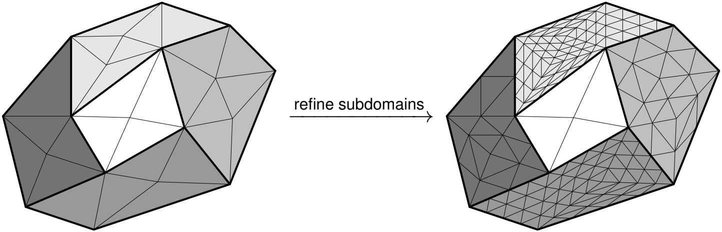
Approach Discretization Strategies

How to discretize the corresponding spaces? See [Wohlmuth, 2001].



For each triangulation $\mathcal{T}_{h,i}$ of Ω_i define

$$X_{h,i} := S^1_{h,i}(\mathcal{T}_{h,i}) = \left\{ u \in \mathcal{C}(\Omega_i) \mid u_{|_T} \in \mathcal{P}^1(T) \quad \forall\, T \in \mathcal{T}_{h,i} \right\}$$

and set

$$X_h := \prod_{i=1}^{N} \left(X_{h,i} \cap H^1_{0,\partial\Omega_i \cap \Gamma_D}(\Omega_i) \right)$$

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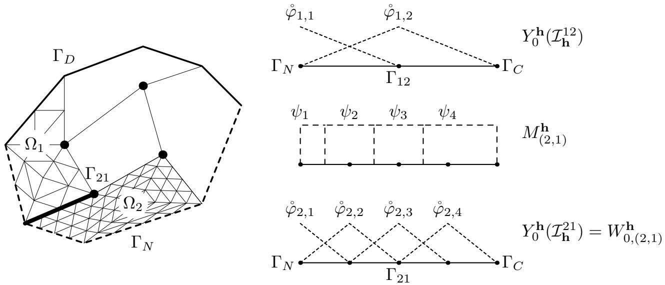

Define $\mathcal{M} := \left\{ m = (k,l) \mid \Gamma_{kl} = \partial\Omega_k \cap \partial\Omega_l \neq \emptyset \text{ with } ''\mathcal{T}_{h,k} \text{ finer than } \mathcal{T}''_{h,l} \right\}$.
For each interface Γ_m define

$$M_{h,m} := S^0_{h,m}(\mathcal{I}'_{h,m}) = \left\{ \lambda \in L_2(\Gamma_m) \mid \lambda_{|E} \in \mathcal{P}^0(E) \quad \forall\, E \in \mathcal{I}'_{h,m} \right\}$$

and set

$$M_h := \prod_{m \in \mathcal{M}} M_{h,m}$$

Approach Discretization Strategies

Assume $\tilde{u}_h := u_h + u_{h,D}$ with $u_h \in X_h$.
We obtain the following modified discrete variational problem.

Transformed Nonlinear Primal Hybrid Variational Formulation

Find $(u_h, \lambda_h) \in X_h \times M_h$, such that

$$a(u_h, v_h) + b(v_h, \lambda_h) = (f, v_h)_{0,\Omega} + (g_N, v_h)_{0,\Gamma_N} - a(u_{h,D}, v_h) \qquad \forall\, v_h \in X_h$$

$$c(u_h, \mu_h) = 0 \qquad \forall\, \mu \in M_h$$

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$$c(u_h, \mu_h) = 0 \qquad \forall\, \mu \in M_h$$

In the discrete setting the bilinear form $b(\cdot,\cdot)$ can be written as

$$b(u_h, \mu_h) := -\sum_{i=1}^{N} \left(u_{h,i}, \mu_h\right)_{0,\partial\Omega_i} = \sum_{m\in\mathcal{M}} \left(u_{h,k} - u_{h,l}, \mu_h\right)_{0,\Gamma_m}$$

$$= \sum_{m\in\mathcal{M}} \left([\![u_h]\!]_{\Gamma_m}, \mu_h\right)_{0,\Gamma_m}.$$

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The same can be done for the linear form $c(\cdot, \cdot)$.

$$\begin{aligned} c(u_h, \mu_h) &:= -\sum_{i=1}^{N} \left(\kappa_i^{-1}(u_{h,i} + u_{h,D,i}), \mu_h\right)_{0,\partial\Omega_i} \\ &= \sum_{m \in \mathcal{M}} \left(\kappa_k^{-1}(u_{h,k} + u_{h,D,k}) - \kappa_l^{-1}(u_{h,l} + u_{h,D,l}), \mu_h\right)_{0,\Gamma_m} \\ &= \sum_{m \in \mathcal{M}} \left([\![\kappa^{-1}(u_h + u_{h,D})]\!]_{\Gamma_m}, \mu_h\right)_{0,\Gamma_m}. \end{aligned}$$

Approach Discretization Strategies

Assume $\tilde{u}_h := u_h + u_{h,D}$ with $u_h \in X_h$.
We obtain the following modified discrete variational problem.

Transformed Nonlinear Primal Hybrid Variational Formulation

Find $(u_h, \lambda_h) \in X_h \times M_h$, such that

$$a(u_h, v_h) + b(v_h, \lambda_h) = (f, v_h)_{0,\Omega} + (g_N, v_h)_{0,\Gamma_N} - a(u_{h,D}, v_h) \qquad \forall\, v_h \in X_h$$

$$c(u_h, \mu_h) = 0 \qquad \forall\, \mu \in M_h$$

To solve the nonlinear problem, we apply Newton's method and solve a sequence of linear problems.

Approach Discretization Strategies

Linearized Formulation

For a given $w_h \in X_h$ find $(u_h, \lambda_h) \in X_h \times M_h$, such that

$$a(u_h, v_h) + b(v_h, \lambda_h) = (f, v_h)_{0,\Omega} + (g_N, v_h)_{0,\Gamma_N} - a(u_{h,D}, v_h)$$
$$c'[w_h](u_h, \mu_h) = c'[w_h](w_h, \mu_h) - c(w_h, \mu_h)$$

for all $(v_h, \mu_h) \in X_h \times M_h$.

The linearized bilinear form $c'[\cdot](\cdot, \cdot)$ is given by

$$c'[w_h](u_h, \mu_h) := \sum_{m \in \mathcal{M}} \left(\kappa_k^{-1\,\prime}\left(w_{h,k} + u_{h,D,k}\right) u_{h,k} - \kappa_l^{-1\,\prime}\left(w_{h,l} + u_{h,D,l}\right) u_{h,l}, \mu_h\right)_{0,\Gamma_m}$$
$$= \sum_{m \in \mathcal{M}} \left(\left[\!\left[\kappa^{-1\,\prime}\left(w_h + u_{h,D}\right) u_h\right]\!\right]_{\Gamma_m}, \mu_h\right)_{0,\Gamma_m}.$$

Approach   Discretization Strategies

To obtain solvability of the variational problem we have to show that

$$\sup_{v_h \in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|_X} \geq \gamma_b \|\mu_h\|_{M_h} \qquad \text{for all } \mu_h \in M_h,$$

$$\sup_{v_h \in X_h} \frac{c'[w_h](v_h, \mu_h)}{\|v_h\|_X} \geq \gamma_c \|\mu_h\|_{M_h} \qquad \text{for all } \mu_h \in M_h,$$

and

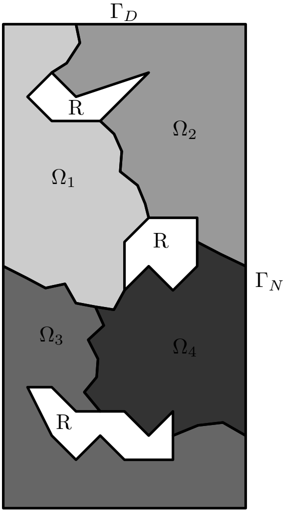
$$\sup_{v_h \in \mathrm{Ker} B} \frac{a(u_h, v_h)}{\|v_h\|_X} \geq \gamma_a \|u_h\|_X \qquad \text{for all } u_h \in \mathrm{Ker} C'[w_h],$$

$$\sup_{v_h \in \mathrm{Ker} C'[w_h]} a(u_h, v_h) > 0 \qquad \text{for all } u_h \in \mathrm{Ker} B,$$

with positive constants γ_b, γ_c and γ_a. See [Nicolaides, 1982].

Outline

Numerical Example



- Consider the instationary Richards equation.
- Domain:

$$\overline{\Omega} = [0,1] \times [0,2] \subset \mathbb{R}^2$$

- Soil types:
  Ω_1 ...sand, Ω_2 ...sandy loam,
  Ω_3 ...loam, Ω_4 ...sand
- Boundary conditions:

$$h_D(\mathbf{x}, t) = \begin{cases} \kappa_1^{-1}(-0.5\,(1-t)) & \mathbf{x} \in \Gamma_{D,1}, t < 1 \\ \kappa_2^{-1}(-0.5\,(1-t)) & \mathbf{x} \in \Gamma_{D,2}, t < 1 \\ 0.0 & \mathbf{x} \in \Gamma_D, t \geq 1 \end{cases}$$

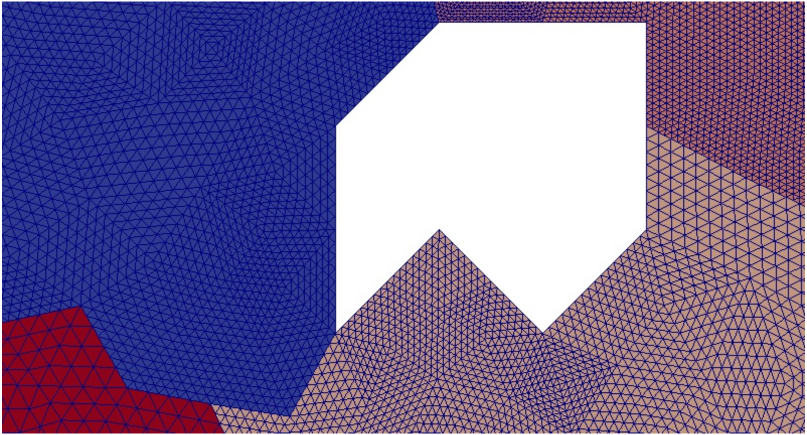
$$h_N(\mathbf{x}, t) = 0.0 \qquad \mathbf{x} \in \partial\Omega \setminus \Gamma_D$$

- Time discretization parameter:
  - Timestep : $\tau = 0.02$
  - Timesteps : $T = 1500$

Matthias Gsell, Institute of Computational Mathematics

Numerical Example

Triangulation of the four domains (zoomed in cutout).



VIDEO : Solution

Outline

Outlook

Summary

- Transform quasiliner PDEs to linear PDEs
- Apply discretization and linearization methods
- Numerical example

Outlook

- Numerical analysis of the discrete linearized SPP
- Convergence analysis
- Preconditioners

Outlook

Summary

- Transform quasiliner PDEs to linear PDEs
- Apply discretization and linearization methods
- Numerical example

Outlook

- Numerical analysis of the discrete linearized SPP
- Convergence analysis
- Preconditioners

Thank you for your attention!

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