Schwarz preconditioning of high order edge elements type discretisations for the time-harmonic Maxwell's equations

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The second order time-harmonic Maxwell's equation

Time-harmonic Maxwell's equation for the electric field **E**:

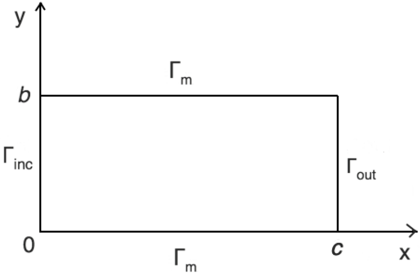
$$\nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right) + \kappa\mathbf{E} = 0, \quad \kappa = i\omega\sigma - \omega^2\varepsilon$$

- ω - angular frequency,
- σ - conductivity of the medium,
- ε - electric permittivity,
- μ - magnetic permeability.

The waveguide problem

Numerical simulation of the waveguide problem:

- rectangular waveguide with perfectly conducting walls,
- a two-dimensional computational domain: $\Omega = [0, c] \times [0, b]$.



The waveguide problem

Boundary value problem:

- metallic boundary conditions on the waveguide walls Γ_m,
- impedance boundary conditions at the waveguide entrance Γ_{inc},
- impedance boundary conditions at the exit Γ_{out}.

$$\begin{cases} \kappa \mathbf{E} + \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{E} \right) = 0, \text{ in } \Omega, \\ \mathbf{E} \times \mathbf{n} = 0, \text{ on } \Gamma_m, \\ \nabla \times \mathbf{E} \times \mathbf{n} - i\tilde{\omega} \mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = g^{inc}, \text{ on } \Gamma_{inc}, \\ \nabla \times \mathbf{E} \times \mathbf{n} - i\tilde{\omega} \mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = g^{out}, \text{ on } \Gamma_{out}, \end{cases}$$

where $\tilde{\omega} = \omega\sqrt{\varepsilon\mu}$ is the wavenumber.

Considering μ constant, the variational formulation of the problem is:

$$\int_\Omega \Big[\mu\kappa \mathbf{E}\cdot\mathbf{v} + (\nabla\times\mathbf{E})\cdot(\nabla\times\mathbf{v})\Big]\,dx - \int_{\Gamma_{\text{inc}}\cup\Gamma_{\text{out}}} i\tilde{\omega}(\mathbf{E}\times\mathbf{n})\cdot(\mathbf{v}\times\mathbf{n})\,d\sigma = \int_{\Gamma_{\text{inc}}} g^{\text{inc}}\cdot\mathbf{v}\,d\sigma + \int_{\Gamma_{\text{out}}} g^{\text{out}}\cdot\mathbf{v}\,d\sigma \quad \forall\mathbf{v}\in V,$$

where

$$V = \{\mathbf{v}\in H(\text{curl},\Omega), \mathbf{v}\times\mathbf{n} = 0 \text{ on } \Gamma_{\text{m}}\}.$$

($H(\text{curl},\Omega)$ - square integrable functions whose curl is also square integrable)

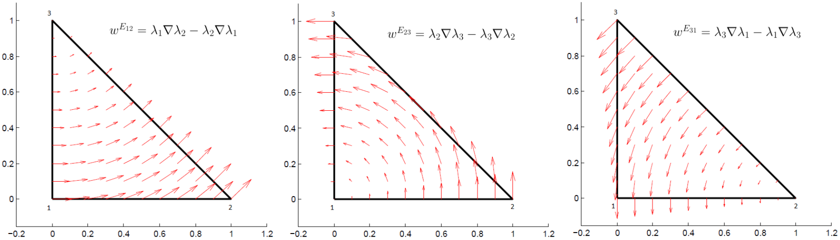
Low order edge finite elements

Finite element discretization: $V_h \subset H(\mathrm{curl}, \Omega)$, triangulation $\mathcal{T}_h$ of Ω.

The simplest $V_h \subset H(\mathrm{curl}, \Omega)$ is given by the low order edge finite elements: the local *basis functions* are *associated with* the *edges* $E = \{l, m\}$ of a given triangle T of $\mathcal{T}_h$:

$$w^E = \lambda_l \nabla \lambda_m - \lambda_l \nabla \lambda_m,$$

(the λ_l are the *barycentric coordinates*). They are *vector* functions!



[Nédélec, Mixed finite elements in $\mathbb{R}^3$, *Numerische Mathematik*, 35, 1980]

Low order edge finite elements

$$E = \{l, m\}, \quad w^E = \lambda_l \nabla \lambda_m - \lambda_l \nabla \lambda_m$$

Properties of w^E

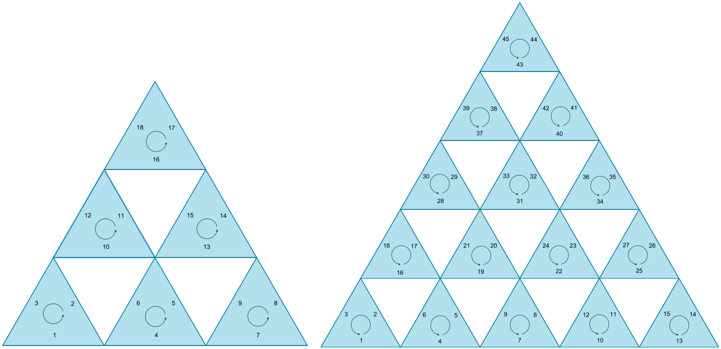
- the unknown degrees of freedom (*dofs*) are circulations (measurable *physical* quantities!) along the mesh edges,
- they ensure tangential continuity across interfaces, allowing discontinuities of the normal component,

$\Rightarrow$ particularly suited for the approximation of electric fields.

We want to maintain these properties for *high order edge elements*.

High order edge elements: construction

Inside each triangle of the mesh (a *big triangle*), we consider an increasing number of *small triangles* (principal lattice).
Polynomial degree $k+1=3$ and $k+1=5$:



Dofs (and the corresponding basis functions) are *associated with* each small edge of each small triangle.

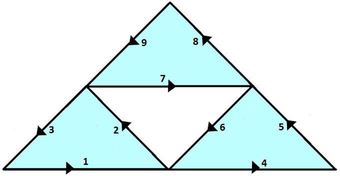
Basis functions of degree $k + 1$ (order k)

For all big edges E, and for all multi-indices $\mathbf{k} = (k_1, k_2, k_3)$ of weight $k = k_1 + k_2 + k_3$, we define:

$$w^e = \lambda^{\mathbf{k}} w^E, \qquad \text{where } \lambda^{\mathbf{k}} = \lambda_1^{k_1} \lambda_2^{k_2} \lambda_3^{k_3}.$$

The couple $e = \{\mathbf{k}, E\}$ can be associated with a small edge.

[Rapetti, Bossavit, Whitney forms of higher degree, *SIAM J. Num. Anal.*, 47(3), 2009]



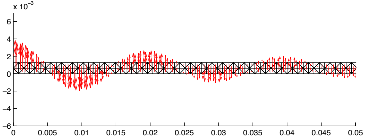
Example:
$k = 1$ ($\rightarrow$ polynomial degree 2)
$\lambda^{e_1} = \lambda_1 w^{E_1}$, $\lambda^{e_2} = \lambda_1 w^{E_2}$, $\lambda^{e_3} = \lambda_1 w^{E_3}$,
$\lambda^{e_4} = \lambda_2 w^{E_1}$, $\lambda^{e_5} = \lambda_2 w^{E_2}$, $\lambda^{e_6} = \lambda_2 w^{E_3}$,
$\lambda^{e_7} = \lambda_3 w^{E_1}$, $\lambda^{e_8} = \lambda_3 w^{E_2}$, $\lambda^{e_9} = \lambda_3 w^{E_3}$.

They are linearly dependent (2 ways to treat this)

Numerical results

Test case:

- exact solution: $\mathbf{E} = (0, e^{-\sqrt{\mu\kappa}x})$;
- dimensions of the waveguide: $c = 0.0502\,\text{m}$, $b = 0.00127\,\text{m}$;



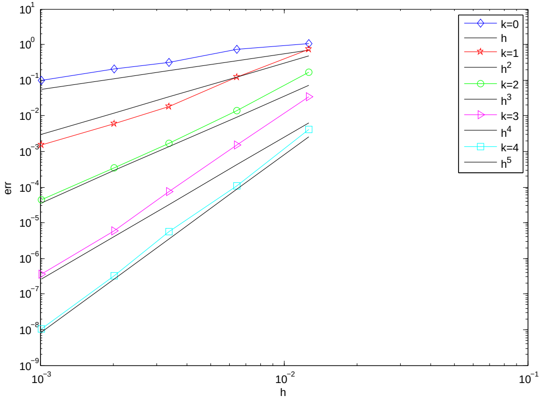
- $\varepsilon = \varepsilon_0 = 8.85 \cdot 10^{-12}\,\text{F/m}$, $\mu = \mu_0 = 1.26 \cdot 10^{-6}\,\text{H/m}$ and $\sigma = 0.15\ \text{S/m}$;
- 3 high frequencies $\omega_1 = 75\,\text{GHz}$, $\omega_2 = 95\,\text{GHz}$, $\omega_3 = 110\,\text{GHz}$;
- elements orders $k = 0, 1, 2, 3, 4$ and five discretization triangle diameters $h_i, i = 1, \ldots, 5$
  ($h_1 = 1.2614 \cdot 10^{-2}\,\text{m}$ is the biggest one and each time we divide it by two).

Remark

With the same number of dofs we get a remarkably smaller error using higher order elements:
e.g. if we choose $k = 1, h = h_5$ or $k = 3, h = h_4$
the errors are respectively 10^{-3} and $6 \cdot 10^{-6}$.

Numerical results: convergence order

- If we fix k and we vary the mesh size h, the convergence to the exact solution is achieved with an order of accuracy equal to $k+1$ w.r.t. h.



- The matrix of the linear system is ill conditioned
  $\Rightarrow$ use a *domain decomposition preconditioner*.

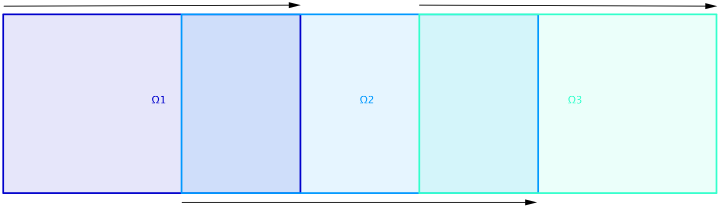
Schwarz preconditioning of high order edge elements

Additive Schwarz preconditioner for the GMRES method:

$$M^{-1} = M_{\mathrm{AS}}^{-1} = \sum_{i=1}^{N} R_i^T A_i^{-1} R_i,$$

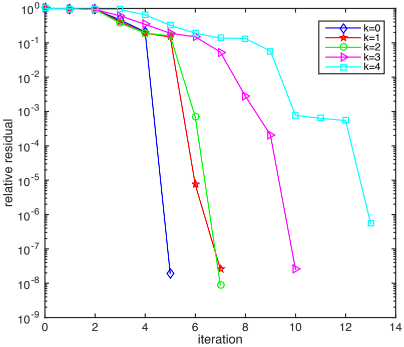
where A_i are the local *subproblems* matrices (impedance conditions as transmission conditions between the subdomains).

Stripwise decomposition into subdomains:



Two subdomains case

$\Omega_1 = (0, c/2 + h_5)$, $\Omega_2 = (c/2 - h_5, c)$, $\omega = \omega_2 = 95\,\text{GHz}$, vary the order k



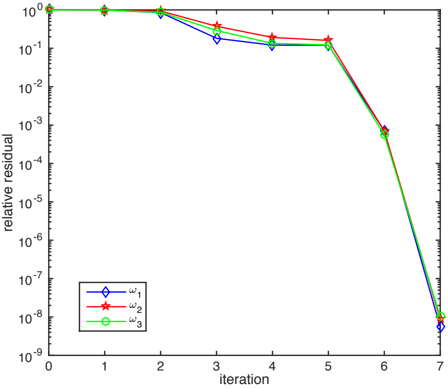
GMRES tol $= 10^{-6}$

With the preconditioner the eigenvalues stay far away from 0 (e.g. $k = 4$: $\min|\lambda| = 0.90$ vs $1.6 \cdot 10^{-3}$).

k	Ndofs	Nits NP	Nits P
0	450	156	5
1	1412	521	7
2	2286	1008	7
3	4872	1963	10
4	7370	3148	13

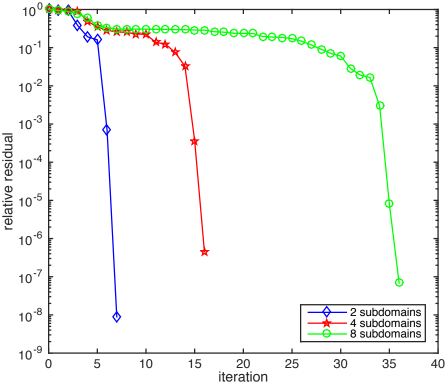
Two subdomains case

$k = 2$, vary the angular frequency ω (Ndofs = 2286)



ω (GHz)	Nits NP	Nits P
75	1014	7
95	1008	7
110	1009	7

$k = 2$, $\omega = \omega_2$, vary the number of subdomains



Nsub	Niter	min$|\lambda|$	max$|\lambda|$
2	7	0.95	10.7
4	16	0.29	10.7
8	36	0.19	13

Ongoing work and outlook

Recently I introduced the high order edge elements (for $k = 1$, in 3D) as a *new finite element* in FreeFem++:

- a (free) domain specific language (DSL) specialised for solving BVPs with variational methods;
- performances close to a low level language;
- simple: just need to write the variational formulation of your BVP;
- linked with many libraries (UMFPACK, Metis, MPI, MUMPS, ...).

[Hecht, FreeFem++, *Numerical Mathematics and Scientific Computation*, LJLL, UPMC]

Not an easy task: *dual basis* needed (and be careful with the *orientation* of the edges)!

Ongoing work and outlook

We want to treat:

- general decompositions of the domain,
- different optimized transmission conditions,
- coarse spaces,
- heterogeneous media,
- three dimensional geometries.