Domain Decomposition Preconditioners for Isogeometric Discretizations

Luca F. Pavarino, Università di Milano, Italy

Lorenzo Beirao da Veiga, Università di Milano, Italy
Durkbin Cho, Dongguk University, Seoul, South Korea
Simone Scacchi, Università di Milano, Italy
Olof B. Widlund, Courant Institute, NYU, USA
Stefano Zampini, KAUST, Saudi Arabia

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IGA Motivations

Resolve the mismatch between CAD and FEM representations in engineering computing practice:

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IGA stiffness matrices very ill-conditioned ($\approx p^{2d+2}4^{pd}$ [Gahalaut et al. 2014]) $\rightarrow$ good preconditioners very much needed

Some IGA DD references

IGA very active emerging field, growing literature, see e.g.
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But few references for IGA iterative solvers are available:

- Overlapping Schwarz [Beirão da Veiga, Cho, LFP, Scacchi, SINUM 2012, CMAME 2013]
- BDDC [Beirão da Veiga, Cho, LFP, Scacchi, M3AN 2013], [Beirão da Veiga, LFP, Scacchi, Widlund, Zampini, SISC 2014]
- IETI [Kleiss, Pechstein, Juttler, Tomar, CMAME 2012]
- Multigrid [Gahalaut, Kraus, Tomar, CMAME 2012], [Takacs et al. 2015]
- BPX [Buffa, Harbrecht, Kunoth, Sangalli, CMAME 2013]
- ...

Notations for B-splines

- $\widehat{\Omega} := (0,1) \times (0,1)$ 2D parametric space.
- Knot vectors
  $\{\xi_1 = 0, \ldots, \xi_{n+p+1} = 1\}, \ \{\eta_1 = 0, \ldots, \eta_{m+q+1} = 1\}$,
  generate a mesh of rectangular elements in parametric space
- 1D basis functions N_i^p, M_j^q, $i = 1, ..., n$, $j = 1, ..., m$ of degree p and q, respectively, are defined from the knot vectors

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2D B-spline space:

$$\widehat{\mathcal{S}}_h = \text{span}\{B_{i,j}^{p,q}(\xi, \eta), \ i = 1, \ldots, n, j = 1, \ldots, m\}$$

Analogously in 3D

Notations for NURBS

- 1D NURBS basis functions of degree p are defined by

$$R_i^p(\xi) = \frac{N_i^p(\xi)\omega_i}{w(\xi)},$$

where $w(\xi) = \sum_{\hat{i}=1}^{n} N_{\hat{i}}^p(\xi)\omega_{\hat{i}} \in \widehat{\mathcal{S}}_h$ is a fixed weight function

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- 2D NURBS basis functions in parametric space $\widehat{\Omega} = (0,1)^2$

$$R_{i,j}^{p,q}(\xi,\eta) = \frac{B_{i,j}^{p,q}(\xi,\eta)\omega_{i,j}}{w(\xi,\eta)},$$

with $w(\xi,\eta) = \sum_{\hat{i}=1}^{n}\sum_{\hat{j}=1}^{m} B_{\hat{i},\hat{j}}^{p,q}(\xi,\eta)\omega_{\hat{i},\hat{j}}$ fixed weight function,
$\omega_{i,j} = (\mathbf{C}_{i,j}^{\omega})_3$ and $\mathbf{C}_{i,j}$ a mesh of $n \times m$ control points

Define the geometrical map $\mathbf{F} : \widehat{\Omega} \to \Omega$ given by

$$\mathbf{F}(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,j}^{p,q}(\xi, \eta) \mathbf{C}_{i,j}.$$

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Space of NURBS scalar fields on a single-patch domain Ω (NURB region) is the span of the *push-forward* of 2D NURBS basis functions (as in isoparametric approach)

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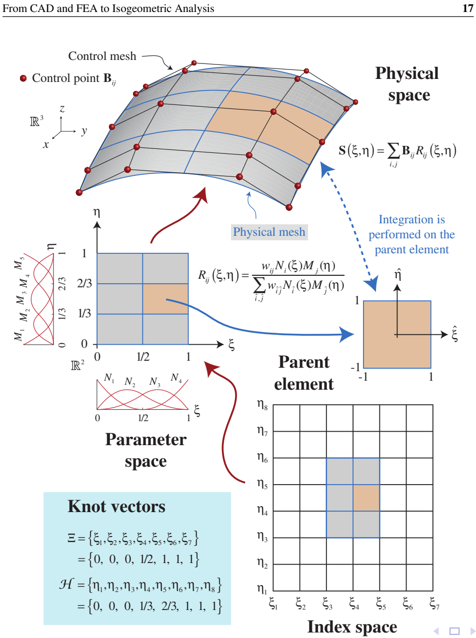
The image of the elements in the parametric space are elements in the physical space. The physical mesh on Ω is therefore

$$\mathcal{T}_h = \{\mathbf{F}((\xi_i, \xi_{i+1}) \times (\eta_j, \eta_{j+1})),\ i = 1, \ldots, n+p,\ j = 1, \ldots, m+q\},$$

where the empty elements are not considered.

(Fig. 1.9 from Cottrell et al., Wiley, 2009)



Model scalar elliptic problem and IGA

Find $u \in V$ such that $a(u,v) = \int_\Omega fvdx \quad \forall v \in V,$ with

- bilinear form $a(u,v) = \int_\Omega \rho \nabla u \nabla v dx$ with $0 < \rho_{min} \leq \rho(x) \leq \rho_{max}$ for all $x \in \Omega \subset \mathbb{R}^d$, a bounded and connected CAD domain

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$$= \text{span}\{B_{i,j}^{p,q}(\xi, \eta),\ i = 2, \ldots, n-1, j = 2, \ldots, m-1\})$$

Elasticity and Stokes considered later.

1D decomposition in parameter space

Nonoverlapping subdomains:

$$\overline{\widehat{I}} = [0,1] = \overline{\bigcup_{k=1,..,N} \widehat{I}_k}, \quad \widehat{I}_k = (\xi_{i_k}, \xi_{i_{k+1}})$$

characteristic subdomain size $H \approx H_k = \mathrm{diam}(\widehat{I}_k)$

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Overlapping subdomains: $\forall \xi_{i_k}$ choose an index s_k (strictly increasing in k) with $s_k < i_k < s_k + p + 1$, so that $supp(N^p_{s_k})$ intersects both $\widehat{I}_{k-1}$ and $\widehat{I}_k$. Then define

$$\widehat{I}'_k = \bigcup_{N^p_j \in \widehat{V}_k} \mathrm{supp}(N^p_j) = (\xi_{s_k - r}, \xi_{s_{k+1}+r+p+1})$$

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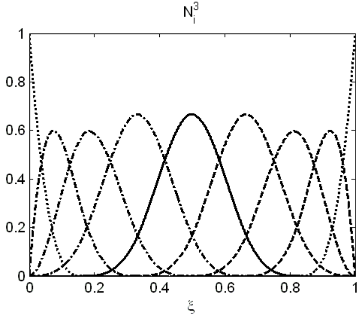
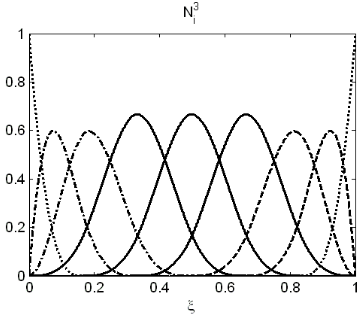
Local subspaces: $\widehat{V}_k = \mathrm{span}\{N^p_j(\xi),\ s_k - r \leq j \leq s_{k+1} + r\}$

1D example with 2 subdomains, 9 basis functions

Subdomains: $\widehat{I}_1 = (0, 1/2)$ and $\widehat{I}_2 = (1/2, 1)$,
Subspaces: $\widehat{V}_1$ and $\widehat{V}_2$



$r = 0$



$r = 1$

$2r + 1$ = number of common basis functions among adjacent subdomains.

1D coarse space

a) Nested coarse space: define a (open) coarse knot vector

$$\boldsymbol{\xi}_0 = \{\xi_1^0 = 0, ..., \xi_{N_c+p+1}^0 = 1\}$$

corresponding to the coarse mesh of subdomains $\widehat{I}_k$. Then

$$\widehat{V}_0 := \mathrm{span}\{N_i^{0,p}(\xi),\ i = 2, ..., N_c - 1\}$$

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b) Non-nested coarse space (standard piecewise linear, $p = 1$):

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2D B-spline decomposition

2D (3D analogous) extension by tensor product:

$$\widehat{I}_k = (\xi_{i_k}, \xi_{i_{k+1}}), \quad \widehat{I}_l = (\eta_{j_l}, \eta_{j_{l+1}}), \quad \widehat{\Omega}_{kl} = \widehat{I}_k \times \widehat{I}_l, \quad \widehat{\Omega}'_{kl} = \widehat{I}'_k \times \widehat{I}'_l.$$

Define local B-spline subspaces:

$$\widehat{V}_{kl} = \left[\mathrm{span}\{ B_{i,j}^{p,q}, \begin{array}{c} s_k - r \leq i \leq s_{k+1} + r, \\ \bar{s}_l - r \leq j \leq \bar{s}_{l+1} + r \end{array} \} \right]^d,$$

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and a coarse B-spline space

$$\widehat{V}_0 = \left[\mathrm{span}\{ \overset{\circ}{B}{}_{i,j}^{p,q}, \begin{array}{l} i = 1, \dots, N_c, \\ j = 1, \dots, M_c \end{array} \} \right]^d,$$

with coarse basis functions $\overset{\circ}{B}{}_{i,j}^{p,q}(\xi,\eta) := N_i^{0,p}(\xi) M_j^{0,q}(\eta)$ (nested)
or $\overset{\circ}{B}{}_{i,j}^{p,q}(\xi,\eta) := N_i^{0,1}(\xi) M_j^{0,1}(\eta)$ (non-nested)

NURBS decomposition in physical space

The subdomains in physical space are defined as the image of the subdomains in parameter space with respect to the mapping $\mathbf{F}$:

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with $\overset{\circ}{R}{}^{p,q}_{i,j} := \overset{\circ}{B}{}^{p,q}_{i,j} / w$ the coarse NURBS basis functions

Overlapping Additive Schwarz (OAS) preconditioners

Given embedding operators
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in matrix form: $\mathbf{T}_{OAS} = \mathbf{B}_{OAS}^{-1} \mathcal{A}$, where $\mathbf{B}_{OAS}^{-1}$ is the OAS prec.

$$\mathbf{B}_{OAS}^{-1} = R_0^T \, A_0^{-1} \, R_0 + \sum_{k=1}^{N} \sum_{l=1}^{M} R_{kl}^T \, A_{kl}^{-1} \, R_{kl}.$$

OAS convergence rate bound:

The condition number of the 2-level additive Schwarz preconditioned isogeometric operator $\mathbf{T}_{OAS}$ is bounded by

$$\kappa_2(\mathbf{T}_{OAS}) \leq C \left(1 + \frac{H}{\gamma}\right),$$

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Scalar proof in: Beirão da Veiga, Cho, LFP, Scacchi. *Overlapping Schwarz methods for Isogeometric Analysis*. SINUM 2012

Compressible elasticity: Beirão da Veiga, Cho, LFP, Scacchi, *Isogeometric Schwarz preconditioners for linear elasticity systems*. CMAME 2013.

Open problems:

- DD theory in p and k,

- extension to other (non-Galerking) IGA variants: IGA collocation (nodal), IGA DG (see work in U. Langer's group)

Numerical results for scalar elliptic pbs.

- 2D and 3D model elliptic problems on both parametric (reference square or cube) and physical domains, zero rhs, Dirichlet or mixed b.c.

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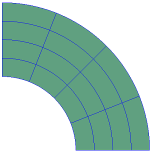
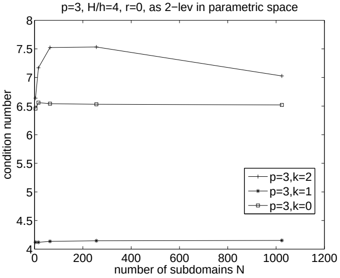
- the domain is decomposed into N overlapping subdomains of characteristic size H and overlap index r
- discrete systems solved by PCG with isogeometric Schwarz preconditioner $\mathbf{B}_{OAS}$, with zero initial guess and stopping criterion a 10^{-6} reduction of the relative PCG residual

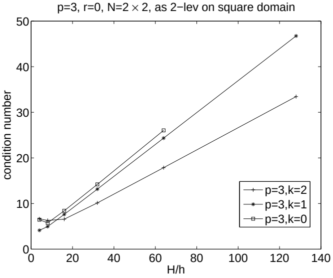


2D Ring domain, NURBS with $p = 3, k = 2$
1- and 2-level OAS preconditioner with $r = 0$

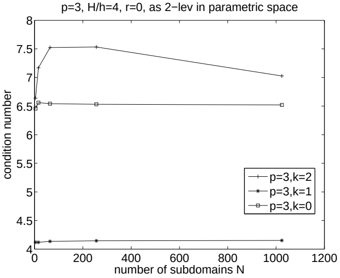
	N	$1/h = 8$ κ_2	it.	$1/h = 16$ κ_2	it.	$1/h = 32$ κ_2	it.	$1/h = 64$ κ_2	it.	$1/h = 128$ κ_2	it.
1-level OAS	2×2	7.69	14	13.07	17	25.10	21	49.49	30	98.47	41
	4×4			18.54	22	39.42	29	81.28	41	165.02	58
	8×8					65.75	38	146.45	54	307.67	78
	16×16							255.98	73	5.75e2	106
	32×32									1.02e3	146
2-level OAS	2×2	7.30	14	6.98	14	11.44	17	20.58	22	38.97	30
	4×4			8.12	18	10.62	20	19.60	23	37.72	32
	8×8					8.41	19	13.92	21	29.88	27
	16×16							8.32	19	15.50	22
	32×32									8.34	19

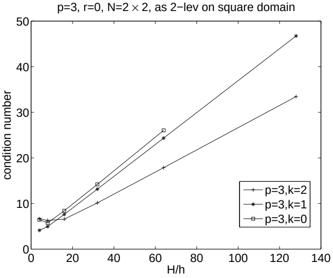
square domain



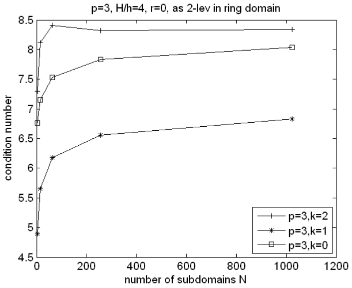


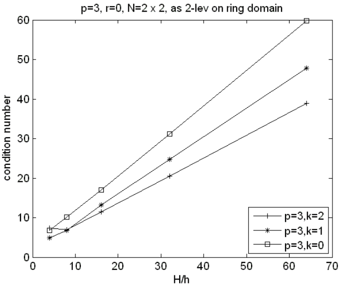
square domain





2D ring domain

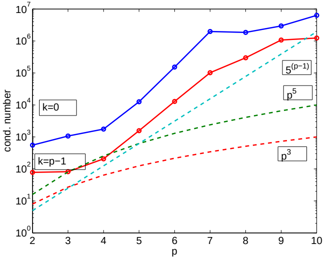




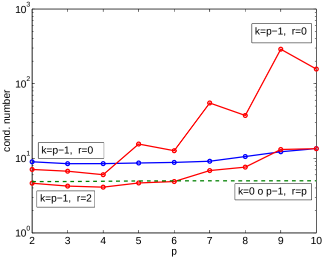
Square domain, $1/h = 32$, $N = 2 \times 2$, $H/h = 16$								
	$k = p - 1$					$k = 0$		
	no prec.	2-level OAS				no prec.	2-level OAS	
p		$r = 0$	$r = 2$	$r = 4$	$r = p$		$r = 0$	$r = p$
2	78.12	7.08	4.63	4.11	4.63	554.89	8.98	4.87
3	82.10	6.71	4.24	4.32	4.18	1.07e+3	8.46	4.88
4	206.71	6.02	4.10	4.29	4.29	1.76e+3	8.47	4.92
5	1.57e+3	15.52	4.67	4.61	4.76	1.26e+4	8.65	4.97
6	1.29e+4	12.64	4.88	4.66	4.79	1.53e+5	8.80	4.98
7	1.02e+5	55.09	6.84	5.21	4.99	1.98e+6	9.13	4.99
8	2.99e+5	37.43	7.61	5.35	4.98	1.86e+6	10.55	4.98
9	1.07e+6	289.61	13.12	6.62	4.99	2.96e+6	12.23	4.99
10	1.24e+6	156.85	13.44	6.20	4.99	6.34e+6	13.48	4.99

Square domain, $1/h = 32$, $N = 2 \times 2$, $H/h = 16$								
	$k = p - 1$					$k = 0$		
	no prec.	2-level OAS				no prec.	2-level OAS	
p		$r = 0$	$r = 2$	$r = 4$	$r = p$		$r = 0$	$r = p$
2	78.12	7.08	4.63	4.11	4.63	554.89	8.98	4.87
3	82.10	6.71	4.24	4.32	4.18	1.07e+3	8.46	4.88
4	206.71	6.02	4.10	4.29	4.29	1.76e+3	8.47	4.92
5	1.57e+3	15.52	4.67	4.61	4.76	1.26e+4	8.65	4.97
6	1.29e+4	12.64	4.88	4.66	4.79	1.53e+5	8.80	4.98
7	1.02e+5	55.09	6.84	5.21	4.99	1.98e+6	9.13	4.99
8	2.99e+5	37.43	7.61	5.35	4.98	1.86e+6	10.55	4.98
9	1.07e+6	289.61	13.12	6.62	4.99	2.96e+6	12.23	4.99
10	1.24e+6	156.85	13.44	6.20	4.99	6.34e+6	13.48	4.99

no prec.

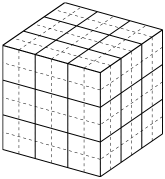


2-level OAS



3D tests: OAS scalability

3D cubic domain, $H/h = 4, p = 3, k = 2$
2-level OAS preconditioner with $r = 0, 1$



N	$r=0$: $\kappa_2 = \lambda_{MAX}/\lambda_{min}$	it.	$r=1$: $\kappa_2 = \lambda_{MAX}/\lambda_{min}$	it.
$2 \times 2 \times 2$	$18.60 = 8.20/0.44$	21	$10.05 = 8.78/0.87$	19
$3 \times 3 \times 3$	$18.80 = 8.26/0.44$	24	$11.92 = 9.63/0.81$	21
$4 \times 4 \times 4$	$19.66 = 8.29/0.42$	25	$12.74 = 9.84/0.77$	22
$5 \times 5 \times 5$	$19.46 = 8.30/0.43$	25	$13.23 = 9.92/0.75$	23
$6 \times 6 \times 6$	$19.52 = 8.31/0.43$	25	$13.40 = 9.99/0.75$	23

3D ring: $16 \times 16 \times 8$, $N = 4 \times 4 \times 2$, $H/h = 4$, $r = 1$, $p = 3$, $k = 2$

central jump

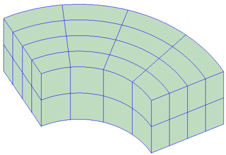
1	1	1	1
1	ρ	ρ	1
1	ρ	ρ	1
1	1	1	1

2nd layer: the same

random mix

10^{-3}	10^{2}	10^{-4}	10^{2}
10^{1}	10^{-1}	10^{0}	10^{4}
10^{-2}	10^{3}	10^{2}	10^{-4}
10^{0}	10^{4}	10^{-3}	10^{1}

2nd layer: reciprocal



3D ring: $16 \times 16 \times 8$, $N = 4 \times 4 \times 2$, $H/h = 4$, $r = 1$, $p = 3$, $k = 2$

central jump

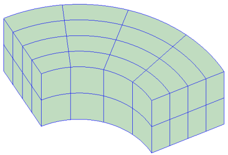
1	1	1	1
1	ρ	ρ	1
1	ρ	ρ	1
1	1	1	1

2nd layer: the same

random mix

10^{-3}	10^{2}	10^{-4}	10^{2}
10^{1}	10^{-1}	10^{0}	10^{4}
10^{-2}	10^{3}	10^{2}	10^{-4}
10^{0}	10^{4}	10^{-3}	10^{1}

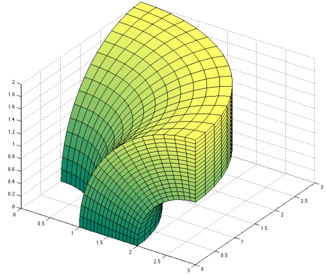
2nd layer: reciprocal



ρ	no prec. $\kappa_2 = \frac{\lambda_{MAX}}{\lambda_{min}}$	it.	1-level OAS $\kappa_2 = \frac{\lambda_{MAX}}{\lambda_{min}}$	it.	2-level OAS $\kappa_2 = \frac{\lambda_{MAX}}{\lambda_{min}}$	it.
central jump						
10^{-4}	$1.42e7 = \frac{6.00e{-1}}{4.24e{-8}}$	7258	$61.65 = \frac{8.00}{1.30e{-1}}$	33	$11.75 = \frac{8.78}{7.49e{-1}}$	22
10^{-2}	$1.11e5 = \frac{6.00e{-1}}{5.41e{-6}}$	873	$61.61 = \frac{8.00}{1.30e{-1}}$	36	$12.15 = \frac{8.78}{7.23e{-1}}$	25
1	$543.38 = \frac{7.94e{-1}}{1.46e{-3}}$	101	$65.82 = \frac{8.00}{1.22e{-1}}$	41	$13.92 = \frac{8.89}{6.39e{-1}}$	26
10^{2}	$1.15e5 = \frac{50.30}{4.39e{-4}}$	1030	$682.26 = \frac{8.00}{1.17e{-2}}$	40	$12.03 = \frac{8.93}{7.42e{-1}}$	23
10^{4}	$1.48e7 = \frac{5.01e3}{3.38e{-4}}$	8279	$6.09e4 = \frac{8.00}{1.31e{-4}}$	49	$12.11 = \frac{8.93}{7.37e{-1}}$	22
random mix						
	$3.26e9 = \frac{6.56e3}{2.01e{-6}}$	$> 10^4$	$30.91 = \frac{8.00}{2.59e{-1}}$	25	$10.70 = \frac{8.67}{8.10e{-1}}$	17

OAS 3D parallel results (PhD thesis of Federico Marini, Univ. of Milan)

3D hose domain
4096 procs.
(FERMI BG/Q)
20^3 local mesh
33.7 M dofs
central jump test
NURBS, $p = 3, \kappa = 2$



ρ	no prec. it.	no prec. $\kappa = \lambda_{max}/\lambda_{min}$	1-level OAS it.	1-level OAS $\kappa = \lambda_{max}/\lambda_{min}$	2-level OAS it.	2-level OAS $\kappa = \lambda_{max}/\lambda_{min}$
10^{-2}	5432	$5.7e5 = \frac{0.07}{1.18e-7}$	140	$578.5 = \frac{8.00}{1.38e-2}$	47	$51.7 = \frac{8.15}{0.158}$
10^4	$\geq 10^4$	$4.8e7 = \frac{347.50}{7.19e-6}$	268	$2.9e6 = \frac{8.0}{2.72e-6}$	66	$131.5 = \frac{8.35}{6.35e-2}$
10^6	$\geq 10^4$	$8.5e7 = \frac{34750.08}{4.06e-4}$	290	$2.9e8 = \frac{8.00}{2.72e-8}$	69	$159.1 = \frac{8.53}{5.36e-2}$

Convection-diffusion problem on $\Omega =$ unit cube,

$$-\epsilon \Delta u + b \nabla u = f \quad in \quad \Omega, \qquad u = 0 \quad on \quad \partial\Omega$$

with $\epsilon = 10^{-2}$, $b = [3, 2, 1]^T$, SUPG stabilization (FERMI BG/Q)

	$p = 2, \kappa = 1$ OAS		$p = 3, \kappa = 2$ OAS	
N (= procs.)	1-lev	2-lev	1-lev	2-lev
$64 = 4 \times 4 \times 4$	25	33	28	40
$512 = 8 \times 8 \times 8$	30	37	30	36
$1728 = 12 \times 12 \times 12$	57	40	47	41
$4096 = 16 \times 16 \times 16$	95	41	85	42

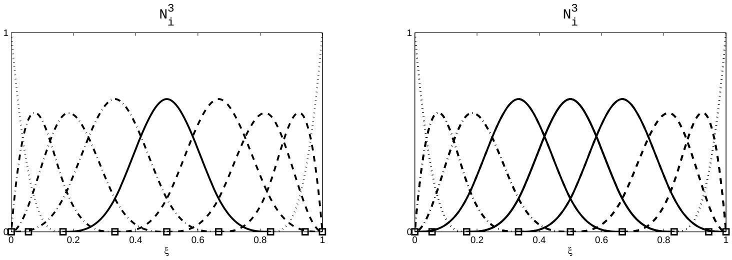
F. Marini, *Overlapping Schwarz preconditioners for isogeometric analysis of convection-diffusion problems.* PhD Thesis, Univ. of Milan, 2015

Parallel library PetIGA by L. Dalcin provides PETSc interface IGA objects

Extension to IGA collocation

Same 1D example with 2 subspaces $\widehat{V}_1$, $\widehat{V}_2$ $\rightarrow$ nodal IGA

Squares = Greville abscissae associated with knot vector ξ



$r = 0$ $\qquad$ $r = 1$

Beirão da Veiga, Cho, LFP, Scacchi, *Overlapping Schwarz preconditioners for isogeometric collocation methods.* CMAME 2014.

Open problems:

- DD Collocation IGA for compressible elasticity,
- DD Collocation IGA for saddle point formulation

Linear Elasticity and Stokes system

- compressible materials, pure displacement formulation OK:

$$2\int_\Omega \mu\epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\, dx + \int_\Omega \lambda \mathrm{div}\mathbf{u}\, \mathrm{div}\mathbf{v}\, dx \;= < \mathbf{F}, \mathbf{v} > \quad \forall \mathbf{v} \in [H^1_{\Gamma_D}(\Omega)]^d$$

λ and μ Lamé constants, $\epsilon(\mathbf{u})$ strain tensor (symmetric gradient)

Linear Elasticity and Stokes system

- compressible materials, pure displacement formulation OK:

$$2\int_\Omega \mu\epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\, dx + \int_\Omega \lambda \mathrm{div}\mathbf{u}\, \mathrm{div}\mathbf{v}\, dx \ = <\mathbf{F},\mathbf{v}> \quad \forall \mathbf{v} \in [H^1_{\Gamma_D}(\Omega)]^d$$

  λ and μ Lamé constants, $\epsilon(\mathbf{u})$ strain tensor (symmetric gradient)

- Almost incompressible elasticity (AIE) and Stokes can suffer from locking phenomena + conditioning degeneration for $\lambda \to \infty$ $(\nu \to 1/2)$. Possible remedy: mixed formulation with displacements (velocities) and pressures:

$$\begin{cases} 2\int_\Omega \mu\epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\, dx & - \int_\Omega \mathrm{div}\mathbf{v}\ p\ dx & = & <\mathbf{F},\mathbf{v}> & \forall \mathbf{v} \in [H^1_{\Gamma_D}(\Omega)]^d \\ -\int_\Omega \mathrm{div}\mathbf{u}\ q\ dx & - \int_\Omega \frac{1}{\lambda} pq\ dx & = & 0 & \forall q \in L^2(\Omega) \\ & & & & (\text{ or } L^2_0(\Omega)) \end{cases}$$

- composite materials with Lamé constants λ_i, μ_i discontinuous across subdomains Ω_i (forming a finite element partition of $\Omega = \bigcup \Omega_i$, with interface $\Gamma = \left(\bigcup_{i=1}^N \partial\Omega_i\right) \setminus \Gamma_D$):

$$\begin{cases} 2\sum_{i=1}^{N} \int_{\Omega_i} \mu_i \ \epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) \ dx & - & \int_{\Omega} \mathrm{div}\mathbf{v} \ p \ dx & = & < \mathbf{F}, \mathbf{v} > \\ \quad -\int_{\Omega} \mathrm{div}\mathbf{u} \ q \ dx & - & \sum_{i=1}^{N} \int_{\Omega_i} \frac{1}{\lambda_i} \ pq \ dx & = & 0 \end{cases}$$

- composite materials with Lamé constants λ_i, μ_i discontinuous across subdomains Ω_i (forming a finite element partition of $\Omega = \bigcup \Omega_i$, with interface $\Gamma = \left(\bigcup_{i=1}^{N} \partial\Omega_i\right) \setminus \Gamma_D$):

$$\begin{cases} 2\sum_{i=1}^{N} \int_{\Omega_i} \mu_i \; \epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) \; dx & - & \int_{\Omega} \operatorname{div}\mathbf{v} \; p \; dx & = & < \mathbf{F}, \mathbf{v} > \\ -\int_{\Omega} \operatorname{div}\mathbf{u} \; q \; dx & - & \sum_{i=1}^{N} \int_{\Omega_i} \frac{1}{\lambda_i} \; pq \; dx & = & 0 \end{cases}$$

- Discretization with IGA finite element spaces $V \subset [H^1_{\Gamma_D}(\Omega)]^d$, $Q \subset L^2(\Omega)$, inf-sup stable in mixed case (LBB condition), see Buffa, De Falco, Sangalli, *Int. J. Numer. Meth. Fluids*, 65, 2011
  For example, IGA Taylor-Hood elements:
  displacements: $V^{p,p-2}$ (degree p, regularity $\kappa = p - 2$)
  pressures: $Q^{p-1,p-2}$ (degree $p - 1$, regularity $\kappa = p - 2$)

- Compressible elasticity: OAS preconditioners built as in the scalar case. Theory extended and confirmed by numerical experiments.
- AIE in mixed form: OAS preconditioners now use saddle point local and coarse problems. Theory still open but numerical experiments OK (GMRES replaces PCG).

Beirão da Veiga, Cho, LFP, Scacchi, *Isogeometric Schwarz preconditioners for linear elasticity systems.* CMAME 2013.

Open problems:

- Schwarz theory for saddle point OAS,
- Positive definite reformulation (IGA has $\geq$ continuous pressures)

Pure displacement formulation degenerates when $\nu \longrightarrow 0.5$

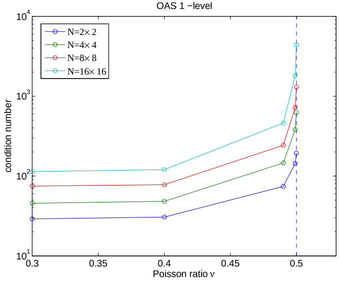
$\hat{\Omega}$ = square, $h = 1/64$, B-splines $p = 3, k = 2$, OAS with $r = 0$

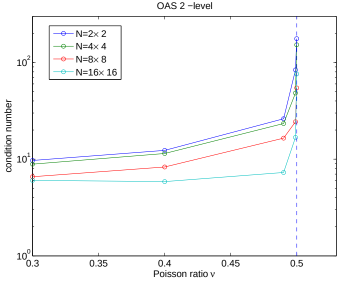
	N	$\nu = 0.3$ κ_2	it.	$\nu = 0.4$ κ_2	it.	$\nu = 0.49$ κ_2	it.	$\nu = 0.499$ κ_2	it.	$\nu = 0.4999$ κ_2	it.
1-level	2×2	29.06	18	30.61	19	74.25	24	143.54	26	193.39	34
	4×4	45.57	27	48.21	31	145.75	43	381.81	51	624.69	59
	8×8	74.84	34	77.83	38	243.04	61	723.01	75	1.3e3	97
	16×16	113.76	46	120.43	52	460.94	89	1.8e3	118	4.4e3	157
2-level	2×2	9.70	16	12.32	17	26.75	20	83.80	25	176.15	31
	4×4	8.88	19	11.45	21	23.38	27	48.33	35	152.21	46
	8×8	6.58	17	8.30	18	16.49	23	24.37	30	54.50	42
	16×16	6.04	18	5.86	18	7.28	19	16.79	25	76.06	46

Pure displacement formulation degenerates when $\nu \longrightarrow 0.5$

$\hat{\Omega}$ = square, $h = 1/64$, B-splines $p = 3$, $k = 2$, OAS with $r = 0$

	N	$\nu = 0.3$ κ_2	it.	$\nu = 0.4$ κ_2	it.	$\nu = 0.49$ κ_2	it.	$\nu = 0.499$ κ_2	it.	$\nu = 0.4999$ κ_2	it.
1-level	2×2	29.06	18	30.61	19	74.25	24	143.54	26	193.39	34
	4×4	45.57	27	48.21	31	145.75	43	381.81	51	624.69	59
	8×8	74.84	34	77.83	38	243.04	61	723.01	75	1.3e3	97
	16×16	113.76	46	120.43	52	460.94	89	1.8e3	118	4.4e3	157
2-level	2×2	9.70	16	12.32	17	26.75	20	83.80	25	176.15	31
	4×4	8.88	19	11.45	21	23.38	27	48.33	35	152.21	46
	8×8	6.58	17	8.30	18	16.49	23	24.37	30	54.50	42
	16×16	6.04	18	5.86	18	7.28	19	16.79	25	76.06	46





While OAS for mixed formulation works well:

2D quarter-ring domain, $E = 6e + 6$ and $\nu = 0.4999$
IGA Taylor-Hood elements: displacements space $p = 3$, $k = 1$
pressure space $p = 2$, $k = 1$
OAS with $r = 1, r_p = 0$

	N	$1/h = 8$ it.	$1/h = 16$ it.	$1/h = 32$ it.	$1/h = 64$ it.	$1/h = 128$ it.
1-level OAS	2×2	23	31	41	55	77
	4×4		44	66	97	179
	8×8			96	193	309
	16×16				320	511
	32×32					900
2-level OAS	2×2	24	26	32	40	54
	4×4		29	33	42	58
	8×8			31	37	49
	16×16				32	38
	32×32					32

OAS robustness when $\nu \to 0.5$

2D quarter-ring domain, $E = 6e+6$

IGA Taylor-Hood elements: displacements space $p = 4,\ k = 2$

pressure space $p = 3,\ k = 2$

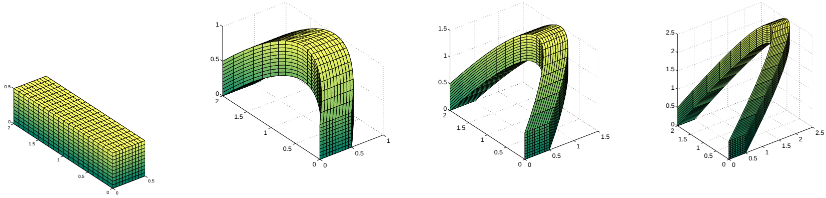
OAS with $r = 1, r_p = 0$

ν	unprec. it.	1-level OAS it.	2-level OAS it.
0.30	123	41	25
0.40	123	46	26
0.49	123	53	28
0.499	123	55	29
0.4999	123	55	29

GMRES iteration counts it. Fixed $h = 1/32$, $N = 4 \times 4$, $H/h = 8$.

Analogous good results for limiting Stokes problem.

3D "boomerang" test, compressible elasticity

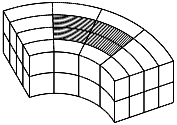


A) B) C) D)

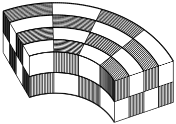
domain	unpreconditioned $\kappa_2 = \frac{\lambda_{max}}{\lambda_{min}}$	unpreconditioned n_{it}	1-level OAS $\kappa_2 = \frac{\lambda_{max}}{\lambda_{min}}$	1-level OAS n_{it}	2-level OAS $\kappa_2 = \frac{\lambda_{max}}{\lambda_{min}}$	2-level OAS n_{it}
overlap $r = 0$						
A	$3.87e+3 = \frac{5.65e+6}{1.46e+3}$	219	$24.85 = \frac{8.00}{0.32}$	30	$22.28 = \frac{8.03}{0.36}$	26
B	$3.74e+3 = \frac{7.55e+6}{2.02e+3}$	292	$117.94 = \frac{8.00}{6.78e-2}$	58	$45.80 = \frac{8.04}{0.18}$	39
C	$8.28e+3 = \frac{1.56e+7}{1.88e+3}$	365	$184.24 = \frac{8.00}{4.34e-2}$	69	$110.38 = \frac{8.06}{7.30e-2}$	54
D	$1.33e+4 = \frac{4.86e+7}{3.65e+3}$	492	$294.18 = \frac{8.00}{2.72e-2}$	76	$223.97 = \frac{8.07}{3.60e-2}$	65
overlap $r = 1$						
A	as above		$14.28 = \frac{9.44}{0.66}$	27	$12.18 = \frac{9.46}{0.78}$	24
B	"		$55.94 = \frac{9.86}{0.18}$	42	$23.23 = \frac{9.95}{0.43}$	31
C	"		$77.05 = \frac{9.87}{0.13}$	51	$49.52 = \frac{9.96}{0.20}$	40
D	"		$118.35 = \frac{9.88}{8.35e-2}$	57	$96.40 = \frac{9.97}{0.10}$	48

Fixed $1/h = 32$, $N = 4 \times 4 \times 2$, $H/h = 4$, $p = 3$, $k = 2$, $\nu = 0.3$, $E = 6e+6$.

3D mixed formulation, 3D quarter-ring domain



A) central jump
$\nu = 0.3$ in white subd.
$\nu \to 0.5$ in gray subd.



B) checkerboard
$\nu = 0.3$ in white subd.
$\nu = 0.4999$ in gray subd.

	ν	unpreconditioned n_{it}	1-level OAS n_{it}	2-level OAS n_{it}
central jump	0.40	88	23	22
	0.49	88	22	23
	0.499	88	22	23
	0.4999	82	30	28
checkerboard ν		89	30	24

IGA Taylor-Hood elements: displacements space $p = 3,\ k = 1$
pressure space $p = 2,\ k = 1$
Fixed $N = 3 \times 3 \times 2$ subdomains, $H/h = 4$
$E = 6e + 6$ everywhere

BDDC (Balancing Domain Decomposition by Constraints) preconditioners

Evolution of Balancing Neumann - Neumann (BNN) prec.
- additive local and coarse problems
- proper choice of primal continuity constraints across the interface of subdomains, as in FETI-DP methods
- dual of FETI-DP preconditioners with same primal space, since both have essentially the same spectrum.

- Dohrmann SISC 25, 2003
- Mandel, Dohrmann, NLAA 10, 2003
- Mandel, Dohrmann, Tezaur, ANM 54, 2005
- FETI-DP: Farhat et al., IJNME 50, 2001
- ...

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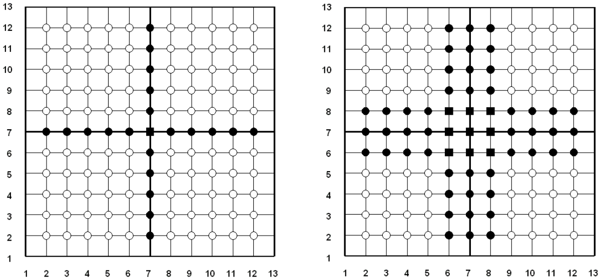
- ...

Recent extension to IGA discretizations of scalar elliptic pbs:
Beirao da Veiga, Cho, LFP, Scacchi, *BDDC preconditioners for Isogeometric Analysis*, M3AS 2013.
Beirao da Veiga, LFP, Scacchi, Widlund, Zampini *Isogeometric BDDC preconditioners with deluxe scaling*, SISC 2014.

Due to the high continuity of IGA basis functions, the Schur complement is associated not just with the geometric interface but with a fat interface: 2×2 example with cubic splines



C^0 splines $\qquad$ C^2 splines

$\circ$ = interior index set

$\bullet$ = interface index set

$\blacksquare$ = vertex (primal) index set

Local Schur complements

Reorder displacements as $(\mathbf{u}_I, \mathbf{u}_\Gamma)$: first interior, then interface.
Then the local spectral element stiffness matrix for subdomain Ω_i is

$$A^{(i)} := \begin{bmatrix} A_{II}^{(i)} & A_{\Gamma I}^{(i)T} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix}$$

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Eliminate interior displacements to obtain local Schur complements

$$S_\Gamma^{(i)} := A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)-1} A_{\Gamma I}^{(i)T}$$

(only implicit elimination, as Schur complements are not needed, only their action on a vector)

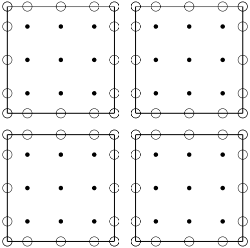
Classical Schur complement:

$$\widehat{S}_\Gamma := \sum_{i=1}^{N} R_\Gamma^{(i)^T} S_\Gamma^{(i)} R_\Gamma^{(i)}$$

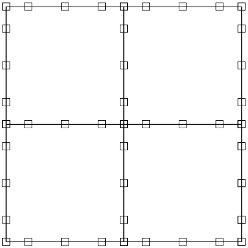
Dual - Primal splitting (BDDC, FETI-DP)

Schematic illustration of the discrete spaces and degrees of freedom in an example with 2×2 subdomains and C^0 (nonfat) interface

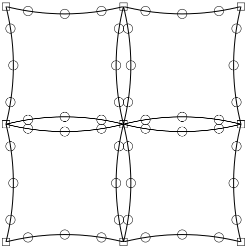
$\bullet$ = interior dofs, $\circ$ = dual dofs, $\square$ = primal dofs.



$V_I \oplus V_\Gamma$



$\widehat{V}_\Gamma$



$\widetilde{V}_\Gamma$

Dual - Primal splitting (BDDC, FETI-DP)

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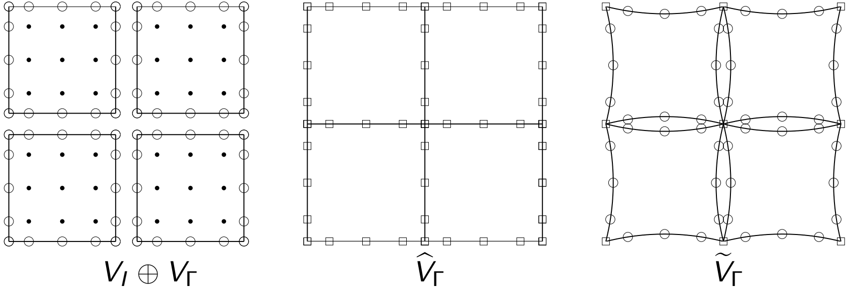
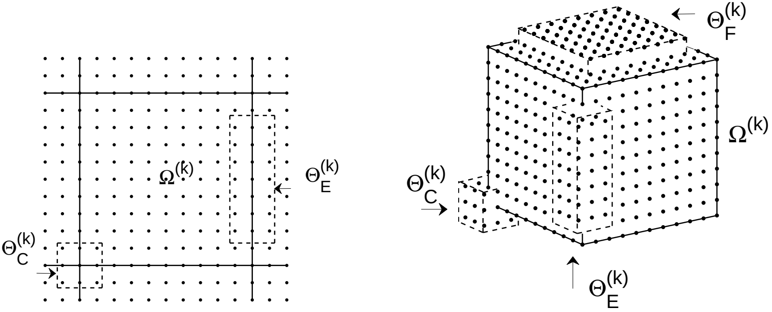


scalar pbs.: u
compressible elasticity: u_1, u_2, u_3
mixed elasticity (and Stokes): u_1, u_2, u_3, p

Examples of equivalent classes with $p = 3, \kappa = 2$

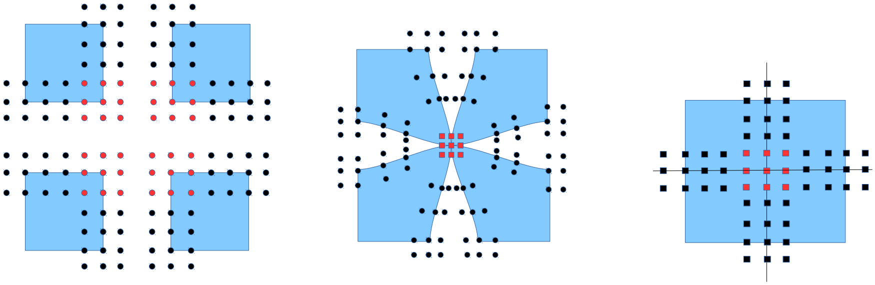


Index space schematic illustration of interface equivalence classes:

- $\Theta_C^{(k)}$ = fat vertex: $(\kappa+1)^2$ knots in 2D, $(\kappa+1)^3$ in 3D
- $\Theta_E^{(k)}$ = fat edge: $(\kappa+1)$ "slim" edges in 2D, $(\kappa+1)^2$ in 3D
- $\Theta_F^{(k)}$ = fat face: $\kappa+1$ slim faces in 3D

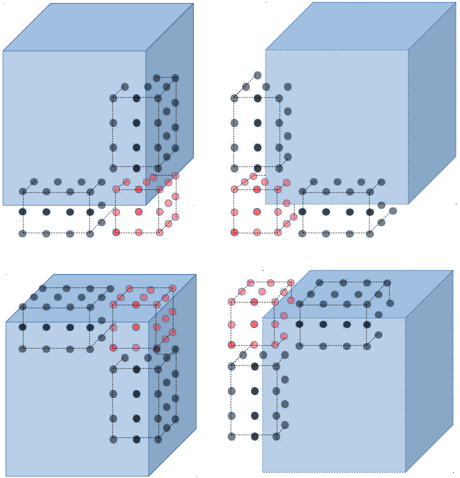
2D example with fat interface for $p = 3$, $\kappa = 2$

circle = dual dofs, squares = primal dofs,
black = edge dofs, red = vertex dofs



fully decoupled V_Γ | partially assembled $\widehat{V}_\Gamma$ | fully assembled $\widetilde{V}_\Gamma$

... analogously in 3D



BDDC preconditioner

Split (fat) interface dofs (displacements, pressures) into dual (Δ) and primal (Π) interface dofs. Local stiffness matrices become

$$A^{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)\,T} & A_{\Pi I}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} & A_{\Pi\Delta}^{(i)\,T} \\ A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} & A_{\Pi\Pi}^{(i)} \end{bmatrix}$$

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The BDDC preconditioner for the Schur complement $\widehat{S}_\Gamma$ is:

$$M^{-1} := \widetilde{R}_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D,\Gamma},$$

where $\widetilde{R}_{D,\Gamma} :=$ the direct sum $R_{\Gamma\Pi} \oplus R_{D,\Delta}^{(i)} R_{\Gamma\Delta}$ with proper restriction/scaling matrices (see later)

and where

$$\widetilde{S}_{\Gamma}^{-1} := R_{\Gamma\Delta}^{T} \left(\sum_{i=1}^{N} \begin{bmatrix} 0 & R_{\Delta}^{(i)^T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^T} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ R_{\Delta}^{(i)} \end{bmatrix} \right) R_{\Gamma\Delta} + \Phi S_{\Pi\Pi}^{-1} \Phi^{T}.$$

and where

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$= \sum_i$ local solvers on each Ω_i with Neumann data on the local edges/faces and with the primal variables constrained to vanish + coarse solve for the primal variables, with coarse matrix

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$$S_{\Pi\Pi} = \sum_{i=1}^{N} R_{\Pi}^{(i)^{T}}\left(A_{\Pi\Pi}^{(i)} - \begin{bmatrix} A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)}\end{bmatrix}\begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^{T}} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)}\end{bmatrix}^{-1}\begin{bmatrix} A_{\Pi I}^{(i)^{T}} \\ A_{\Pi\Delta}^{(i)^{T}}\end{bmatrix}\right) R_{\Pi}^{(i)}$$

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and change of variable matrix Φ

$$\Phi = R_{\Gamma\Pi}^T - R_{\Gamma\Delta}^T \sum_{i=1}^N \begin{bmatrix} 0 & R_\Delta^{(i)^T} \end{bmatrix} \begin{bmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)^T} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{bmatrix}^{-1} \begin{bmatrix} A_{\Pi I}^{(i)^T} \\ A_{\Pi\Delta}^{(i)^T} \end{bmatrix} R_\Pi^{(i)}.$$

BDDC scaling operators

Scaling operator $R_D = D\tilde{R}$, with $D = diag(D^{(j)})$ restores continuity during Krylov iteration and takes into account possible jumps of elliptic coefficient ρ on Γ

Standard scaling: $D^{(j)}$ diagonal with elements

$$\delta_j^\dagger(x_i) = \delta_j(x_i) / \sum_{k \in N_x} \delta_k(x_i), \quad x_i \in \mathbf{W}_\Delta$$

- ρ-scaling: $\delta_j(x_i) = \rho_j(x_i)$
- stiffness scaling: $\delta_j(x_i) = A_{ii}^{(j)}$

Deluxe scaling (Dohrmann- Widlund, DD21): $D^{(j)}$ block diagonal with blocks

$$\Big(\sum_{k \in N_F} S_F^{(k)} \Big)^{-1} S_F^{(j)}, \qquad F = \text{vertex, edge, face of } \Gamma$$

where $S_F^{(j)}$ = principal minor of $S^{(j)}$ associated with the dofs in F

Possible choices of primal constraints

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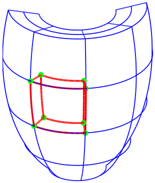
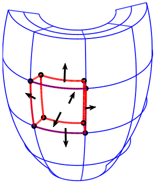
- E: averages of displacements/pressures over each subdomain edge (for u the 2 normals E_a^2 or all 3 E_a^3);
- E_m: first order moments of displacements/pressures over each subdomain edge (for u the 2 normals or all 3);
- F: averages of displacements/pressures over interior of each subdomain face (for u 1 normal average F_a^1 or all 3 F_a^3);



V



$V + F_a^1$

IGA BDDC convergence rate bounds for elliptic pbs.

Theorem

The condition number of the BDDC (and associated FETI - DP) preconditioned isogeometric operator is bounded by

$$\kappa_2(M^{-1}\widehat{S}_\Gamma) \leq C\left(1 + \log^2(\frac{H}{h})\right) \qquad \textit{for } \rho\textit{-scaling and deluxe}$$

$$\kappa_2(M^{-1}\widehat{S}_\Gamma) \leq C\frac{H}{h}\left(1 + \log^2(\frac{H}{h})\right) \qquad \textit{for stiffness scaling}$$

where C is a constant independent of h, H, N (but not of p, k).

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- Beirao da Veiga, Cho, LFP, Scacchi, *BDDC preconditioners for Isogeometric Analysis*. M3AS 2013
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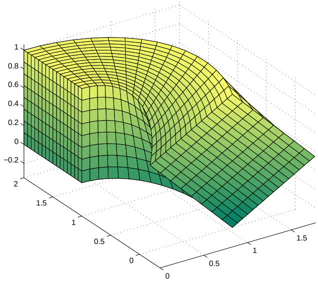
Weak scalability on unit cube, fixed $\kappa = 2, p = 3, H/h = 8$
Parallel tests on FERMI BG/Q with PETSc PCBDDC class (by S. Zampini)

	N	2^3	3^3	4^3	5^3	6^3	7^3	8^3	9^3	10^3
		Deluxe scaling								
$\widehat{V}_\Pi^C$	k_2	8.96	8.38	8.44	8.38	8.35	8.35	8.35	8.36	8.35
	n_{it}	20	21	23	24	23	23	24	24	24
$\widehat{V}_\Pi^{CE}$	k_2	2.06	2.01	1.98	1.98	1.98	1.98	1.98	1.98	1.98
	n_{it}	10	11	11	10	10	10	10	10	10
$\widehat{V}_\Pi^{CEF}$	k_2	1.42	1.40	1.41	1.40	1.40	1.40	1.40	1.40	1.40
	n_{it}	8	8	8	8	8	8	8	8	8
		Stiffness scaling								
$\widehat{V}_\Pi^C$	k_2	20.09	19.24	19.16	19.16	19.16	19.16	19.16	19.16	19.17
	n_{it}	26	33	38	39	39	39	39	39	39
$\widehat{V}_\Pi^{CE}$	k_2	6.04	6.08	6.08	6.10	6.09	6.10	6.09	6.10	6.10
	n_{it}	21	22	22	22	22	23	22	23	22
$\widehat{V}_\Pi^{CEF}$	k_2	6.04	6.08	6.08	6.10	6.09	6.10	6.09	6.10	6.10
	n_{it}	21	22	22	22	22	23	22	23	22

Weak scalability on twisted domain, fixed $\kappa = 2, p = 3, H/h = 6$



	N	2^3	3^3	4^3	5^3	6^3
		Deluxe scaling				
$\widehat{V}_{\Pi}^{C}$	k_2	3.94	5.72	6.87	7.47	7.83
	n_{it}	11	15	20	21	23
$\widehat{V}_{\Pi}^{VE}$	k_2	1.67	1.81	1.85	1.86	1.92
	n_{it}	9	10	10	10	10
$\widehat{V}_{\Pi}^{CEF}$	k_2	1.42	1.58	1.66	1.72	1.76
	n_{it}	8	9	9	9	9
		Stiffness scaling				
$\widehat{V}_{\Pi}^{C}$	k_2	9.39	11.07	12.97	13.87	14.39
	n_{it}	24	29	30	31	33
$\widehat{V}_{\Pi}^{CE}$	k_2	8.94	9.21	9.27	9.35	9.38
	n_{it}	24	27	28	28	29
$\widehat{V}_{\Pi}^{CEF}$	k_2	8.94	9.21	9.27	9.35	9.38
	n_{it}	24	27	28	28	29

BDDC deluxe robustness with respect to jump discontinuities in the diffusion coefficient ρ,
fixed $h = 1/32, p = 3$, C^0 continuity at the interface, $4 \times 4 \times 4$ subdomains

	ρ	central jump k_2	central jump n_{it}	checkerboard k_2	checkerboard n_{it}	random mix k_2	random mix n_{it}
$\widehat{V}_{\Pi}^{C}$	10^{-4}	117.37	44	—	—	—	—
	10^{-2}	118.40	44	—	—	—	—
	1	134.04	48	134.04	48	134.04	48
	10^{2}	137.15	50	102.11	43	126.53	47
	10^{4}	137.40	52	104.31	44	123.63	46
$\widehat{V}_{\Pi}^{CE}$	10^{-4}	5.33	18	—	—	—	—
	10^{-2}	5.33	18	—	—	—	—
	1	5.27	18	5.27	18	5.27	18
	10^{2}	4.92	16	4.19	16	4.83	16
	10^{4}	4.88	16	4.20	16	4.87	16
$\widehat{V}_{\Pi}^{CEF}$	10^{-4}	1.98	10	1.98	10	—	—
	10^{-2}	1.99	10	1.99	10	—	—
	1	2.05	10	2.05	10	2.05	10
	10^{2}	2.05	10	2.05	10	2.00	10
	10^{4}	2.05	10	2.05	10	2.00	9

deluxe BDDC dependence on p,
2D quarter-ring domain: fixed $h = 1/64$, $N = 4 \times 4$, $\kappa = p - 1$

p	2	3	4	5	6	7	8	9	10
k_2	3.22	2.68	2.41	2.19	2.04	1.91	1.80	1.72	1.62
n_{it}	10	10	9	9	9	8	8	8	9

3D unit cube, fixed $h = 1/24$, $N = 2 \times 2 \times 2$, $\kappa = p - 1$

	p	2	3	4	5	6	7
		Deluxe scaling					
$\widehat{V}_{\Pi}^{C}$	k_2	5.62	4.71	4.39	3.92	5.12	11.15
	n_{it}	12	11	12	14	18	26
$\widehat{V}_{\Pi}^{CE}$	k_2	2.10	1.91	2.03	2.68	4.99	10.92
	n_{it}	10	9	10	12	17	26
$\widehat{V}_{\Pi}^{CEF}$	k_2	1.58	1.45	1.70	2.68	4.99	10.92
	n_{it}	8	8	9	12	17	26

Open problems:

- BDDC, FETI-DP for IGA collocation
- BDDC, FETI-DP for elasticity with IGA Galerkin/collocation

Extension to Elasticity and Stokes problems

- Compressible elasticity: BDDC preconditioners built as in the scalar case. Scalar theory can be extended and is confirmed by numerical experiments.
- AIE in mixed form: BDDC preconditioners now use saddle point local and coarse problems. Theory still open but numerical experiments ok (GMRES replaces PCG).

Open problems:
- AIE positive definite reformulation for IGA ($\geq$ continuous pressures), deluxe scaling?
- extending to IGA the FEM preconditioners in Li and Tu SINUM 2013, IJNME 2013, Kim and Lee CMAME 2012

$S^{(k)}$ = local Schur complement associated to Ω_k

$\mathcal{F}$ = one of the equivalence classes: vertex, edge, or face

Partition $S^{(k)} = \begin{pmatrix} S^{(k)}_{\mathcal{F}\mathcal{F}} & S^{(k)}_{\mathcal{F}\mathcal{F}'} \\ S^{(k)}_{\mathcal{F}'\mathcal{F}} & S^{(k)}_{\mathcal{F}'\mathcal{F}'} \end{pmatrix}$ and define the new Schur complement of Schur complements $\widetilde{S}^{(k)}_{\mathcal{F}\mathcal{F}} = S^{(k)}_{\mathcal{F}\mathcal{F}} - S^{(k)}_{\mathcal{F}\mathcal{F}'} S^{(k)^{-1}}_{\mathcal{F}'\mathcal{F}'} S^{(k)}_{\mathcal{F}'\mathcal{F}}$

Generalized eigenvalue problem V_1

$$S^{(k)}_{\mathcal{F}\mathcal{F}} v = \lambda \widetilde{S}^{(k)}_{\mathcal{F}\mathcal{F}} v. \qquad (1)$$

Given a threshold $\theta \geq 1$:

- select the eigenvectors $\{v_1, v_2, \ldots, v_{N_c}\}$ associated to the eigenvalues of (1) greater than θ,
- perform a BDDC change of basis and make these selected eigenvectors the primal variables.

Adaptive primal spaces by parallel sums

Define the parallel sum of two positive definite matrices A and B as

$$A : B = (A^{-1} + B^{-1})^{-1} \qquad \text{(analog. for } \geq 3 \text{ matrices)}$$

IGA 2D: each fat vertex is shared by 4 subdomains $\Omega_i, \ i = 1, 2, 3, 4$

Generalized eigenvalue problem V_{par}:

Define V_{par} as the parallel sum primal space based on the parallel sum generalized eigenvalue problem

$$\left(S^{(1)}_{\mathcal{F}\mathcal{F}} : S^{(2)}_{\mathcal{F}\mathcal{F}} : S^{(3)}_{\mathcal{F}\mathcal{F}} : S^{(4)}_{\mathcal{F}\mathcal{F}}\right) v = \lambda \left(\widetilde{S}^{(1)}_{\mathcal{F}\mathcal{F}} : \widetilde{S}^{(2)}_{\mathcal{F}\mathcal{F}} : \widetilde{S}^{(3)}_{\mathcal{F}\mathcal{F}} : \widetilde{S}^{(4)}_{\mathcal{F}\mathcal{F}}\right) v$$

Generalized eigenvalue problem V_{mix}:

Define V_{mix} as the mixed primal space based on the mixed parallel sum generalized eigenvalue problem

$$\left(S^{(1)}_{\mathcal{F}\mathcal{F}} : S^{(2)}_{\mathcal{F}\mathcal{F}} : S^{(3)}_{\mathcal{F}\mathcal{F}} : S^{(4)}_{\mathcal{F}\mathcal{F}}\right) v = \lambda \left(\widetilde{S}^{(1)}_{\mathcal{F}\mathcal{F}} + \widetilde{S}^{(2)}_{\mathcal{F}\mathcal{F}} + \widetilde{S}^{(3)}_{\mathcal{F}\mathcal{F}} + \widetilde{S}^{(4)}_{\mathcal{F}\mathcal{F}}\right) v$$

Minimal Vertex primal space V_1: K and H/h dependence

Minimal $N_c = 1$ primal constraint per vertex (turns out to be the average over the fat vertex)

N	$h=1/8$ cond	it.	$h=1/16$ cond	it.	$h=1/32$ cond	it.	$h=1/64$ cond	it.	$h=1/128$ cond	it.
	$p=3,\ k=1$		NURBS, quarter-ring domain							
2×2	1.74	7	2.08	7	2.29	7	2.85	8	3.45	8
4×4			4.34	13	5.91	14	7.59	15	9.42	15
8×8					5.37	15	7.41	18	9.53	21
16×16							5.98	16	8.34	19
32×32									6.31	17
	$p=3,\ k=2$		NURBS, quarter-ring domain							
2×2	1.45	7	2.00	8	2.72	8	3.57	8	4.52	8
4×4			10.06	15	13.90	16	18.66	18	23.92	21
8×8					12.13	24	17.42	27	24.85	32
16×16							12.79	24	18.96	29
32×32									13.04	24

condition number (cond) and iteration counts (it.) as functions of the number of subdomains N and mesh size h.

Minimal vertex primal space V_1: p dependence

NURBS, quarter-ring domain						
	$k = p - 1$		$k = 2$		$k = 1$	
p	cond	it.	cond	it.	cond	it.
2	7.09	14			7.09	14
3	18.66	18	18.66	18	7.59	15
4	233.81	26	19.74	20	8.31	15
5	8417.70	56	22.22	19	9.06	15
6			25.37	21	9.81	16
7			29.05	22	10.52	16
8			33.08	23	11.24	17
9			37.64	24	11.90	17
10			39.89	26	12.59	18

Minimal Vertex - Edge primal space VE_1: K and H/h dependence

$p=3,\ k=2$	NURBS, quarter-ring domain									
	$h=1/8$		$h=1/16$		$h=1/32$		$h=1/64$		$h=1/128$	
N	cond	it.	cond	it.	cond	it.	cond	it.	cond	it.
2×2	1.44	7	1.97	7	2.65	8	3.46	8	4.37	8
4×4			5.09	13	4.65	13	5.31	14	5.99	15
8×8					6.20	17	5.34	15	6.00	16
16×16							6.66	18	5.73	16
32×32									6.83	18

Minimal $N_c = 1$ primal constraint per vertex and $N_c = 1$ per edge

NURBS, quarter-ring domain						
	$k = p - 1$		$k = 2$		$k = 1$	
p	cond	it.	cond	it.	cond	it.
2	2.91	11			2.91	11
3	5.31	14	5.31	14	2.80	11
4	41.17	24	4.85	21	2.88	11
5	1598.65	67	4.77	14	3.00	11
6			4.93	15	3.13	11
7			5.16	16	3.27	12
8			5.67	17	3.40	12
9					3.53	13
10					3.71	13

Adaptive vertex space V_1 with N_c primal constraints

N	$N_c = 1$ ($\theta = 2$) cond	n_{it}	$N_c = 4$ ($\theta = 1.5$) cond	n_{it}
2×2	1.81	7	1.66	8
4×4	12.74	14	6.74	13
8×8	14.74	24	7.48	18
16×16	15.67	26	7.78	18
32×32	16.13	24	7.87	17

a) scalability in N
for fixed $p = 3, \kappa = 2, H/h = 8$

H/h	$N_c = 1$ ($\theta = 2$) cond	n_{it}	$N_c = 4$ ($\theta = 1.5$) cond	n_{it}
4	8.75	12	4.84	12
8	12.74	14	6.74	13
16	17.40	17	8.91	14
32	22.31	18	11.16	15
64	27.49	20	13.50	17

b) H/h dependence
for fixed $p = 3, \kappa = 2, N = 4 \times 4$

p	$N_c = 1$ ($\theta = 2$) cond	n_{it}	($\theta = 1.1$) cond	n_{it}	N_c
2	6.09	13	3.55	11	3
3	17.40	17	5.34	14	5
4	230.9	21	5.74	15	8
5	7545.9	39	12.25	18	10
6	-	-	73.08	31	12

c) p dependence
for fixed $N = 4 \times 4, H/h = 16, \kappa = p - 1$

Mixed space V_{mix} with minimal $N_c = 1$ primal constraints

Even with V_{mix}, the first eigenvector is the average over the fat vertices, but the other eigenvectors (change of basis) change $\rightarrow$ much better performance than with V_1 primal space

Increasing p, 2D quarter-ring domain						
	$k=3$		$k=4$		$k=p-1$	
p	cond	it.	cond	it.	cond	it.
2	n/a	n/a	n/a	n/a	5.54	13
3	n/a	n/a	n/a	n/a	5.50	13
4	6.02	14	n/a	n/a	6.02	14
5	6.14	14	5.77	14	5.77	14
6	6.39	15	5.88	15		
7	6.76	18	6.48	18		
8	8.65	21	12.61	26		
9	13.13	27	23.17	34		
10	19.18	33				

Open problems:

- find good adaptive primal spaces for elasticity (we have fat globs for each component!)

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  - H/h - optimality (OAS), H/h - quasi-optimality (BDDC)
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THANK YOU