A locking-free hybrid DGFEM for nearly incompressible materials

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Introduction

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2. Volume Locking, which is an unpreferable phenomenon.
3. Two kinds of Hybrid DGFEMs (Discontinuous Galerkin Finite Element Methods) are introduced.
   - One of them is locking free, and the other one is not.
4. These facts are shown theoretically and numerically.
5. Conclusion

Plane Strain Problem

- $\underline{u} = [u_1,\ u_2]^T$: two-dimensional displacement of the elastic body.
- The strain tensor $\underline{\underline{\varepsilon}}(\underline{u}) = [\varepsilon_{ij}(\underline{u})]_{ij}$ is given by

  $\varepsilon_{ij}(\underline{u}) = \frac{1}{2}\left(\partial u_i/\partial x_j + \partial u_j/\partial x_i\right)\ (1 \leq i,\ j \leq 2)$.
- We use an underline (resp. double underlines) to denote two dimensional vector (resp. 2×2 matrix) valued functions, operators, and their associated spaces.

Plane Strain Problem

- The isotropic linear elastic stress-strain relation is written by

$$\underline{\sigma}(\underline{u}) = 2\mu\,\underline{\varepsilon}(\underline{u}) + \lambda(\operatorname{div}\underline{u})\,\underline{\delta},$$

where λ and μ are Lamé parameters,

$$\underline{\delta} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- We assume $\lambda > 0$ and $\mu = 1$ in this talk.

Plane Strain Problem

We consider the following plane strain problem:

$$\begin{cases} -\underline{\operatorname{div}}\,\underline{\underline{\sigma}}(\underline{u}) = \underline{f} & \text{in} \quad \Omega, \\ \underline{u} = \underline{0} & \text{on} \quad \partial\Omega, \end{cases}$$

$\underline{f} = [f_1,\ f_2]^T$ is a distributed external body force per unit in-plane area.

Volume Locking

- When the Lamé constant $\lambda\ (> 0)$ is large, the accuracy of FE solutions obtained by using coarse meshes is bad. So we need to use sufficiently fine meshes to obtain satisfactory FE solutions.
- Babuška–Suri(1992) presented a mathematical definition of the volume locking. Our theoretical analysis will be based on it.
- It is well known that P_1 conforming FEM causes a volume locking phenomenon.

A volume locking phenomenon in the conforming P_1 FEM

- Domain $\Omega := (0,\, 1) \times (0,\, 1)$.
- We determine the exact solution $\underline{u}$ by

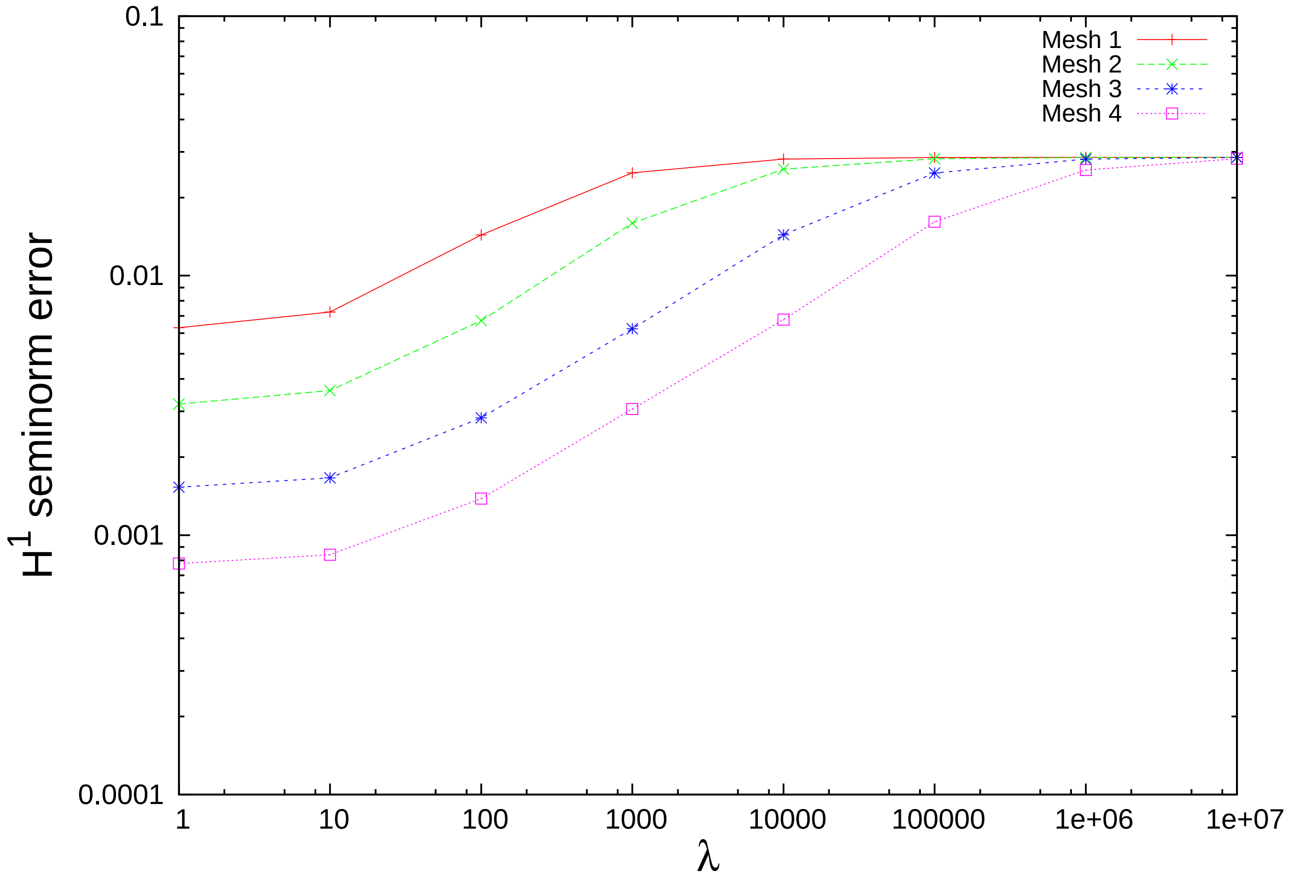
$$
\begin{aligned}
\psi(x) &:= x^2(x-1)^2, \\
\Psi(x_1,\, x_2) &:= -\frac{1}{2}\psi(x_1)\psi(x_2) \quad \text{(stream function)}, \\
\underline{u} &:= \operatorname{rot} \Psi.
\end{aligned}
$$

- The exact solution is independent of λ and satisfies $\operatorname{div} \underline{u} = 0$.
- This test problem is presented in Bercovier–Livne (1979) and Soon–Cockburn–Stolarski (2009).

A volume locking phenomenon in the conforming P_1 FEM

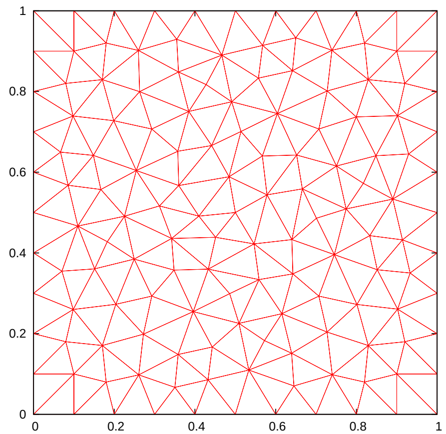
- Let us solve the test problem by P_1 conforming FEM.
- We use 4 meshes which are obtained by dividing each side of Ω into $2^j \times 10$ $(j = 0, 1, \ldots, 3)$ equi-length line segments. To make these meshes, we used Gmsh [15].

Error vs. λ

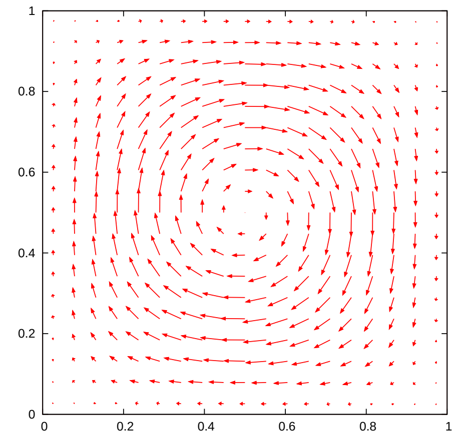


A volume locking phenomenon arising in P_1 conforming FEM

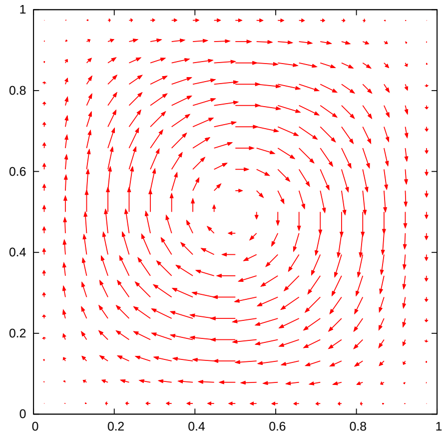
Vector fields



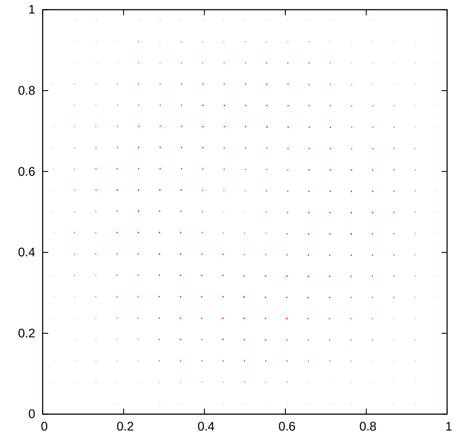
$h = 1/10$



FEM sol. $(\lambda = 1)$

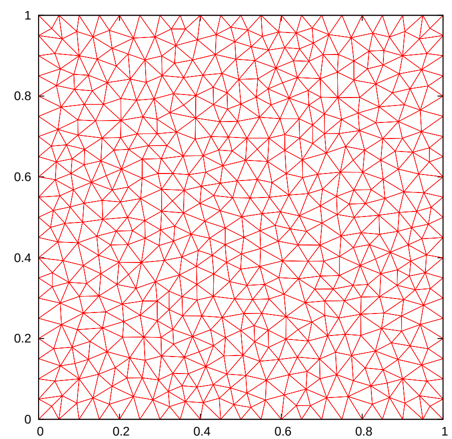


Exact sol. $\underline{u}$

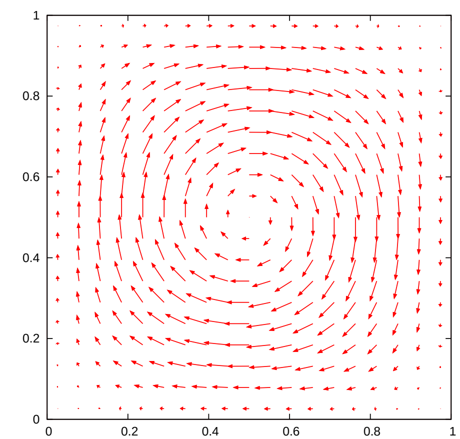


FEM sol. $(\lambda = 5000)$

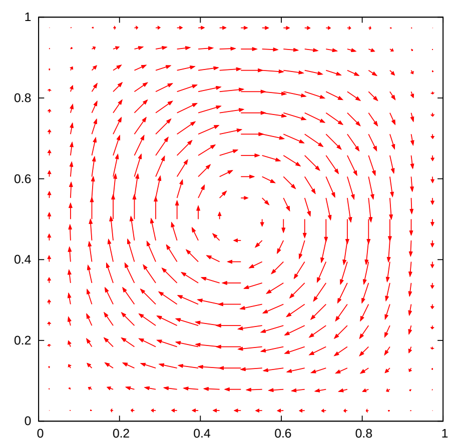
Vector fields



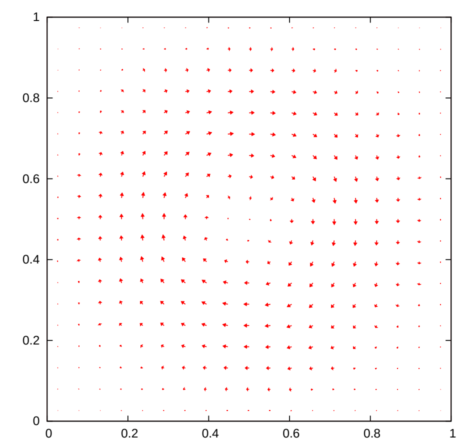
$h = 1/20$



FEM sol. $(\lambda = 1)$

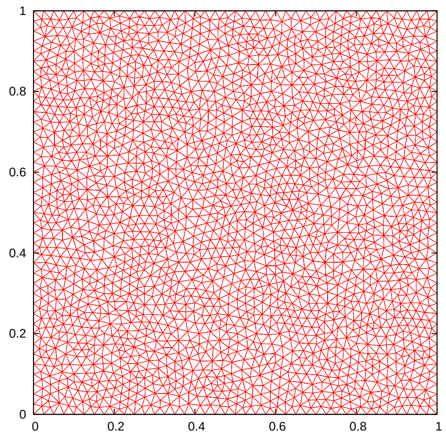


Exact sol. $\underline{u}$

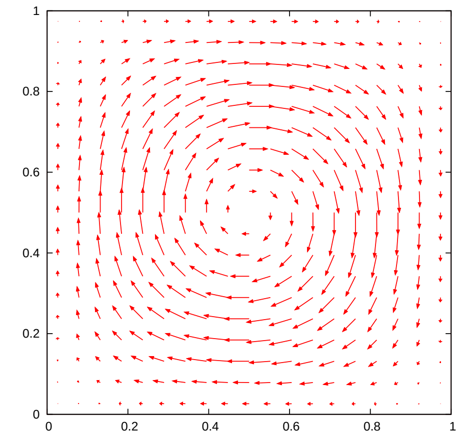


FEM sol. $(\lambda = 5000)$

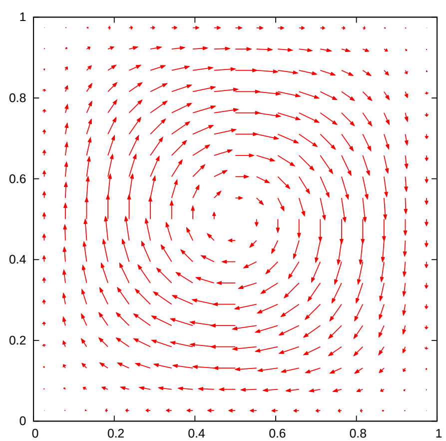
Vector fields



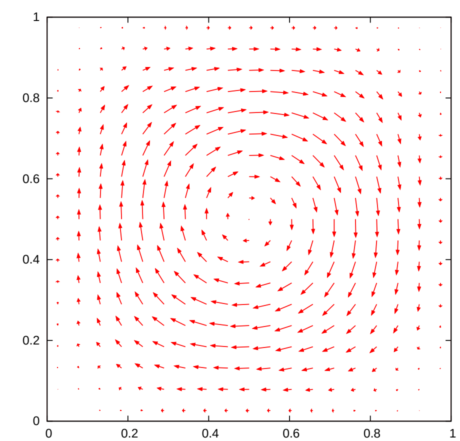
$h = 1/40$



FEM sol. $(\lambda = 1)$

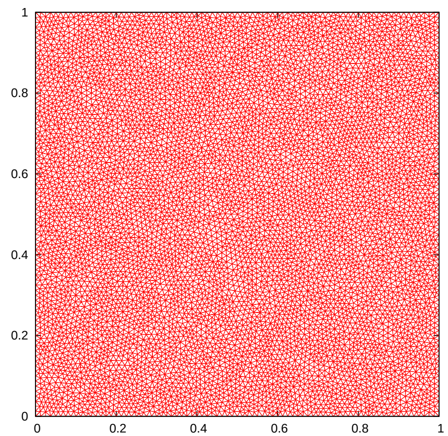


Exact sol. $\underline{u}$

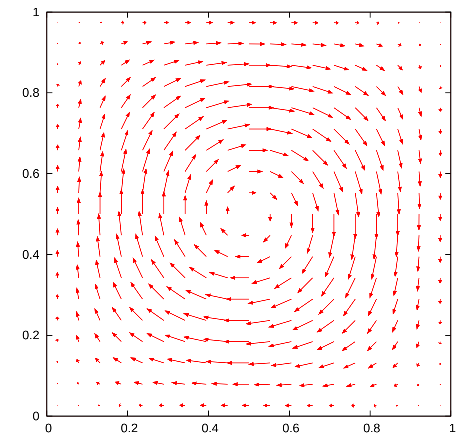


FEM sol. $(\lambda = 5000)$

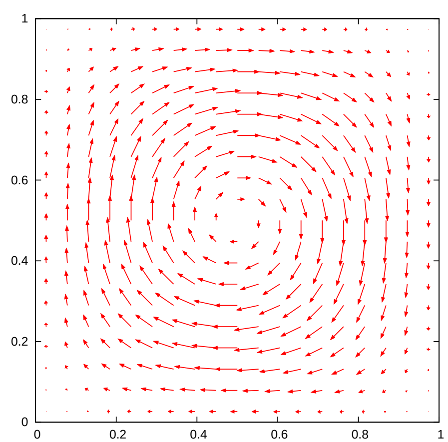
Vector fields



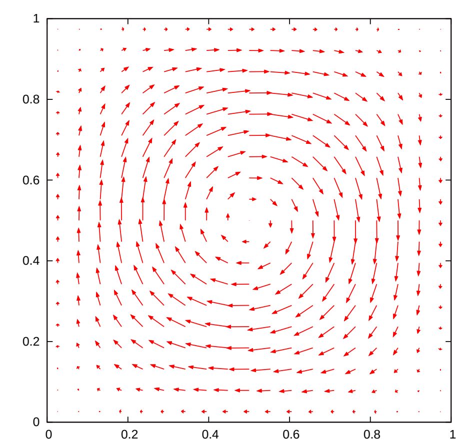
$h = 1/80$



FEM sol. $(\lambda = 1)$



Exact sol. $\underline{u}$



FEM sol. $(\lambda = 5000)$

Remedies for the Volume Locking

- High-order FE
  - Babuška–Suri, 1992
- Mixed methods
  - Arnold–Brezzi–Douglas, 1984
  - Stenberg, 1988
  - Jeon–Sheen, 2013
- Non-conforming FE
  - Brenner–Sung, 1992
- DG
  - Hansbo–Larson, 2002 (not Hybirid type)
  - Wihler, 2004 (not Hybirid type)
  - Soon–Cockburn–Stolarski, 2009 (a hybrid type different from ours)
  - Di Pietro–Nicaise, 2013 (not Hybrid type)

Hybrid DGFEMs

A hybrid version of SIP method

- We consider a hybrid version of SIP (Symmetric Interior Penalty) method, which is called Hybrid DGFEM in this talk.
- The SIP method was first investigated by Wheeler (1978) and Arnold (1982).
- The hybrid version has been investigated by the following authors:
  - ✦ Laplace eq.: Oikawa–Kikuchi (2010)
  - ✦ Linear elasticity eq.: Kikuchi–Ishii–Oikawa (2009)
  - ✦ Convection diffusion eq.: Oikawa (2014)
  - ✦ Stokes eq.: Egger–Waluga (2013)
  - ✦ Rellich-type discrete compactness: Kikuchi (2012)

Weak formulation in our Hybrid DGFEM

- $\mathcal{T}^h$: a triangulation of $\Omega \subset \mathbb{R}^2$.
- We assume that a family of triangulations $\{\mathcal{T}^h\}_{0<h\leq\bar{h}}$ is regular in the sense of Ciarlet.
- $\mathcal{E}^h$: the set of all edges of $\mathcal{T}^h$.
- $\Gamma^h := \bigcup_{e\in\mathcal{E}^h} \bar{e}$, which is called skeleton.

Weak formulation in our Hybrid DGFEM

- Let $\underline{u} (\in \underline{H}^s(\Omega))$ $(s > 3/2)$ be the exact solution of the plane strain problem.
- We denote the trace on the skeleton Γ^h of $\underline{u}$ by $\underline{\hat{u}}$, i.e., $\underline{\hat{u}} := \underline{u}|_{\Gamma^h}$.
- We call $\underline{\hat{u}}$ Numerical Trace (NT) in this talk.
- In Hybrid version, we treat $\underline{u}$ and $\underline{\hat{u}}$ as unknowns.
- We approximate $\underline{u}$ and $\underline{\hat{u}}$ by piecewise linear functions, i.e., we use the following FE spaces:

$$U^h := \prod_{K\in\mathcal{T}^h} P_1(K)$$

(piecewise linear functions on Ω),

$$\widehat{U}^h := \prod_{e\in\mathcal{E}^h} P_1(e)$$

(piecewise linear functions on Γ^h).

Weak formulation in our Hybrid DGFEM

Then $\underline{\boldsymbol{u}} := \{\underline{u},\, \underline{\hat{u}}\}$ satisfies the following weak form

$$a_\eta^h(\underline{\boldsymbol{u}},\, \underline{\boldsymbol{v}}) = \left(\underline{f},\, \underline{v}\right)_\Omega$$

for all $\underline{\boldsymbol{v}} := \{\underline{v},\, \underline{\hat{v}}\} \in \underline{H}^s(\mathcal{T}^h) \times \underline{L}_D^2(\Gamma^h)$.

- broken Sobolev space: $\forall s > 0$,

$$H^s(\mathcal{T}^h) := \left\{v \in L^2(\Omega);\; v|_K \in H^s(K),\, \forall K \in \mathcal{T}^h\right\}.$$

- $(\cdot,\,\cdot)_\Omega$: the standard inner product of $L^2(\Omega)$.
- $L_D^2(\Gamma^h) := \left\{\hat{v} \in L^2(\Gamma^h) \mid \hat{v} = 0 \text{ on } \partial\Omega\right\}$.
- We will define the bilinear form $a_\eta^h(\cdot,\,\cdot)$ on the next sheet.

Bilinear form $a_\eta^h(\cdot,\cdot)$ with penalty parameter $\eta > 0$

$$
\begin{aligned}
a_\eta^h(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}})
:= \sum_{K\in\mathcal{T}^h} \Big[& 2\mu \Big(\underline{\varepsilon}(\underline{u}), \underline{\varepsilon}(\underline{v})\Big)_K + \lambda\,(\operatorname{div}\underline{u}, \operatorname{div}\underline{v})_K \\
& + \underbrace{\Big\langle \underline{\sigma}(\underline{u})\underline{n}, \hat{\underline{v}} - \underline{v} \Big\rangle_{\partial K}}_{\text{Consistency term}} + \underbrace{\Big\langle \hat{\underline{u}} - \underline{u}, \underline{\sigma}(\underline{v})\underline{n} \Big\rangle_{\partial K}}_{\text{Symmetry term}} \Big] \\
& + \underbrace{L^h(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}})}_{\text{Lifting term}} + \underbrace{\eta I^h(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}})}_{\text{Penalty term}}
\end{aligned}
$$

- $(\cdot,\cdot)_K$ and $\langle\cdot,\cdot\rangle_{\partial K}$ are the standard inner products of $L^2(K)$ and $L^2(\partial K)$, respectively.

Lifting term

For each $K \in \mathcal{T}^h$, a local lifting operator:

$$R_i^K : L^2(\partial K) \longrightarrow P_0(K) \quad (i = 1,\, 2)$$

is defined by

$$(R_i^K g,\, \varphi)_K = \langle g,\, \varphi n_i \rangle_{\partial K} \quad \forall g \in L^2(\partial K), \quad \forall \varphi \in P_0(K).$$

- $P_0(K)$: the set of constant functions on K.
- $\underline{n} = [n_1,\, n_2]^T$: the outward unit normal $\underline{n}$ on ∂K.
- Lifting operator R_i^K corresponds to the differential operator $\partial/\partial x_i$.

Lifting term

The lifting operators corresponding to div, ε_{ij}, and $\underline{\varepsilon}$ are defined as follows: for $\underline{g} = [g_1,\, g_2]^T \in \underline{L}^2(\partial K)$,

$$\begin{aligned} R^K_{\mathrm{div}}\underline{g} &:= \sum_{i=1}^{2} R^K_i g_i, \\ R^K_{\varepsilon_{ij}}\underline{g} &:= \frac{1}{2}\left(R^K_i g_j + R^K_j g_i\right) \quad (1 \le i,\, j \le 2), \\ \underline{\underline{R^K_{\underline{\varepsilon}}}}(\underline{g}) &:= \left[R^K_{\varepsilon_{ij}}\underline{g}\right]_{1 \le i,\, j \le 2}. \end{aligned}$$

Lifting term

We finally define

$$L^h(\underline{\boldsymbol{u}},\,\underline{\boldsymbol{v}}) := \sum_{K\in\mathcal{T}^h}\Big[2\mu\left(\underline{\underline{R_\varepsilon^K}}(\underline{\hat{u}}-\underline{u}),\,\underline{\underline{R_\varepsilon^K}}(\underline{\hat{v}}-\underline{v})\right)_K + \lambda\left(R_{\mathrm{div}}^K(\underline{\hat{u}}-\underline{u}),\,R_{\mathrm{div}}^K(\underline{\hat{v}}-\underline{v})\right)_K\Big].$$

(Interior) Penalty term

Bilinear form I^h is defined as follows:
$\forall \underline{\boldsymbol{u}} = \{\underline{u}, \underline{\hat{u}}\}, \ \underline{\boldsymbol{v}} = \{\underline{v}, \underline{\hat{v}}\} \in \underline{H}^1(\mathcal{T}^h) \times \underline{L}^2(\Gamma^h)$,

$$I^h(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}) := \sum_{K \in \mathcal{T}^h} \sum_{e \in \mathcal{E}^K} \frac{1}{|e|} \langle \underline{\hat{u}} - \underline{u}, \underline{\hat{v}} - \underline{v} \rangle_e .$$

- $\mathcal{E}^K$: the set of all edges of K.
- $|e|$: the length of an edge e.
- $\langle \cdot, \cdot \rangle_e$: the standard inner product on $L^2(e)$.

Another bilinear form $b_\eta^h(\cdot,\,\cdot)$

We also consider another bilinear form $b_\eta^h(\cdot,\,\cdot)$ obtained by subtracting the lifting term from a_η^h:

$$b_\eta^h(\underline{\boldsymbol{u}},\,\underline{\boldsymbol{v}}) := a_\eta^h(\underline{\boldsymbol{u}},\,\underline{\boldsymbol{v}}) - L^h(\underline{\boldsymbol{u}},\,\underline{\boldsymbol{v}}).$$

Another bilinear form $b_\eta^h(\cdot,\,\cdot)$

We also consider another bilinear form $b_\eta^h(\cdot,\,\cdot)$ obtained by subtracting the lifting term from a_η^h:

$$b_\eta^h(\underline{\boldsymbol{u}},\,\underline{\boldsymbol{v}}) := a_\eta^h(\underline{\boldsymbol{u}},\,\underline{\boldsymbol{v}}) - L^h(\underline{\boldsymbol{u}},\,\underline{\boldsymbol{v}}).$$

$$\begin{aligned} b_\eta^h(\underline{\boldsymbol{u}},\,\underline{\boldsymbol{v}}) \;=\; \sum_{K\in\mathcal{T}^h} \Big[& 2\mu \left(\underline{\varepsilon}(\underline{u}),\,\underline{\varepsilon}(\underline{v})\right)_K + \lambda\,(\mathrm{div}\,\underline{u},\,\mathrm{div}\,\underline{v})_K \\ & + \underbrace{\left\langle \underline{\sigma}(\underline{u})\underline{n},\,\underline{\hat{v}} - \underline{v}\right\rangle_{\partial K}}_{\text{Consistency term}} + \underbrace{\left\langle \underline{\hat{u}} - \underline{u},\,\underline{\sigma}(\underline{v})\underline{n}\right\rangle_{\partial K}}_{\text{Symmetry term}} \Big] \\ & + \underbrace{\eta I^h(\underline{\boldsymbol{u}},\,\underline{\boldsymbol{v}})}_{\text{Penalty term}} . \end{aligned}$$

What motivates us to exclude the lifting term?

Now let us consider a time-dependent elastic wave equation and boundary conditions:

$$\begin{aligned} \frac{\partial^2 \underline{u}}{\partial t^2} - \underline{\mathrm{div}}\, \underline{\underline{\sigma}}(\underline{u}) &= \underline{f} \quad \text{in } \Omega, \\ \underline{u} &= \underline{0} \quad \text{on } \partial\Omega. \end{aligned}$$

Semi-discrete problem

Its semi-discrete problem can be represented as a differential-algebraic equation:

$$\frac{d^2}{dt^2}\begin{bmatrix} M_{11} & O \\ O & O \end{bmatrix}\begin{bmatrix} \boldsymbol{u}(t) \\ \hat{\boldsymbol{u}}(t) \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}\begin{bmatrix} \boldsymbol{u}(t) \\ \hat{\boldsymbol{u}}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}(t) \\ \boldsymbol{0} \end{bmatrix}.$$

Deleting $\hat{\boldsymbol{u}}$, we can reduce this equation to

$$\frac{d^2}{dt^2} M_{11}\boldsymbol{u}(t) + \left(A_{11} - A_{12}A_{22}^{-1}A_{12}^T\right)\boldsymbol{u}(t) = \boldsymbol{f}(t).$$

Submatrix A_{22}

- To numerically solve

$$\frac{d^2}{dt^2} M_{11}\boldsymbol{u}(t) + \left(A_{11} - A_{12}A_{22}^{-1}A_{12}^T\right)\boldsymbol{u}(t) \;=\; \boldsymbol{f}(t),$$

  we need to compute the following matrix–vector product: $A_{22}^{-1}\vec{v}$.
- If we exclude the lifting term and if we properly choose a basis of $P_1(e)^2$ for each $e \in \mathcal{E}^h$, then A_{22} can be the unit matrix, and hence we do not need to compute $A_{22}^{-1}\vec{v}$.
- If we add the lifting term, then A_{22} is NOT a block diagonal matrix, and hence we have to compute $A_{22}^{-1}\vec{v}$ with much effort.
- NOTE: For steady problems, we can also use another Schur complement matrix: $A_{22} - A_{12}^T A_{11}^{-1} A_{12}$. A_{11} can be the unit matrix.

Hybrid DGFEMs

We consider two types of Hybrid DGFEMs:

1. DG with Lifting term (DG-wL):
   find $\underline{\boldsymbol{u}}^h = \{\underline{u}^h,\, \underline{\hat{u}}^h\} \in \underline{\boldsymbol{V}}^h$ such that

$$a_\eta^h(\underline{\boldsymbol{u}}^h,\, \underline{\boldsymbol{v}}^h) = (\underline{f},\, \underline{v}^h)_\Omega \quad \forall \underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}^h.$$

2. DG without Lifting term (DG-woL):
   find $\underline{\boldsymbol{u}}^h = \{\underline{u}^h,\, \underline{\hat{u}}^h\} \in \underline{\boldsymbol{V}}^h$ such that

$$b_\eta^h(\underline{\boldsymbol{u}}^h,\, \underline{\boldsymbol{v}}^h) = (\underline{f},\, \underline{v}^h)_\Omega \quad \forall \underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}^h.$$

- $\widehat{V}^h := \widehat{U}^h \cap L_D^2(\Gamma^h)$ and $\underline{\boldsymbol{V}}^h := \underline{U}^h \times \underline{\widehat{V}}^h$.
- $L_D^2(\Gamma^h) := \{\hat{v} \in L^2(\Gamma^h) \mid \hat{v} = 0$ on $\partial\Omega\}$.

Hybrid DGFEMs

Our goal is to show the following two facts theoretically and numerically:

1. DG-wL is locking free.
2. DG-woL can not prevent locking phenomena.

Coerciveness

Coerciveness of a_η^h and b_η^h

Proposition 1 *$\exists C > 0$ such that $\forall \eta > 0$, $\forall \lambda > 0$, $\forall h \in (0, \bar{h}]$, and $\forall \underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}^h$,*

$$a_\eta^h(\underline{\boldsymbol{v}}^h, \underline{\boldsymbol{v}}^h) \geq C \min\{1, \eta\} \|\underline{\boldsymbol{v}}^h\|_{\underline{\boldsymbol{V}}^h}^2,$$

where C is independent of λ, h, η, and $\underline{\boldsymbol{v}}^h$.

Proposition 2 *$\exists C > 0$ such that $\forall \eta > \eta_0 := 2C_r(\lambda + 2\mu)$, $\forall \lambda > 0$, $\forall h \in (0, \bar{h}]$, and $\forall \underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}^h$,*

$$b_\eta^h(\underline{\boldsymbol{v}}^h, \underline{\boldsymbol{v}}^h) \geq C \min\{1, \eta\} \|\underline{\boldsymbol{v}}^h\|_{\underline{\boldsymbol{V}}^h}^2,$$

where C is independent of λ, h, η, and $\underline{\boldsymbol{v}}^h$, and C_r will be given below.

Coerciveness of a_η^h and b_η^h

- If we use a_η^h, we take an arbitrary η.
- If we use b_η^h and if we adopt the sufficient condition: $\eta > \eta_0 = 2C_r(\lambda + 2\mu)$, then we have to take $\eta = O(\lambda)$ as $\lambda \longrightarrow \infty$.
- Is it reasonable to use the sufficient condition in practical computations?
- We numerically examine how well η_0 estimates the exact lower bound η_{LB}^h, which is given as follows:

$$\eta_{\mathrm{LB}}^h = \inf\{\eta > 0 \mid b_\eta^h \text{ is coercive}\}.$$

Coerciveness of a_η^h and b_η^h

Here a norm of $\underline{\boldsymbol{V}}^h$ is defined as follows: $\forall\,\{\boldsymbol{v},\,\hat{\boldsymbol{v}}\} \in \underline{\boldsymbol{V}}^h$,

$$\|\{\boldsymbol{v},\,\hat{\boldsymbol{v}}\}\|_{\underline{\boldsymbol{V}}^h}^2 := \sum_{K\in\mathcal{T}^h}\left\{|\boldsymbol{v}|_{H^1(K)}^2 + \sum_{e\in\mathcal{E}^K}\left[\frac{1}{|e|}|\hat{\boldsymbol{v}}-\boldsymbol{v}|_e^2 + |e||\nabla\boldsymbol{v}|_e^2\right]\right\}.$$

Lower bound η_0

The constant C_r in the definition of $\eta_0 := 2C_r(\lambda + 2\mu)$: appears in the following estimate.

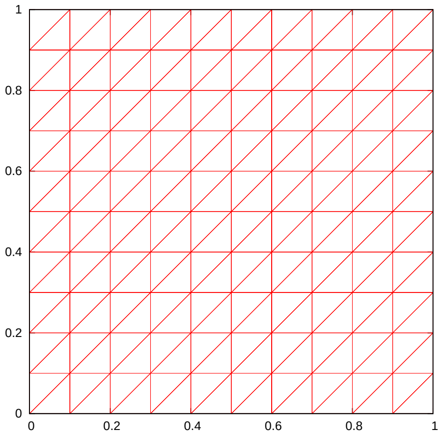
Lemma 1 *There exists a positive constant C_r such that for all $h \in (0, \bar{h}]$, for all $K \in \mathcal{T}^h$, and for all $g \in \prod_{e \in \mathcal{E}^K} P_k(e)$,*

$$\left\| R_i^K g \right\|_K^2 \leq C_r \sum_{e \in \mathcal{E}^K} \frac{1}{|e|} |g|_e^2 \quad (i = 1, 2),$$

where C_r is independent of h, K, and g.

Lower bound η_0

- If K is an isosceles right triangle, we can find that $C_r = 4$.
- In numerical computations below, we use triangulations of Friedrichs–Keller (FK) type as shown in the figure below, whose elements are all isosceles right triangles.



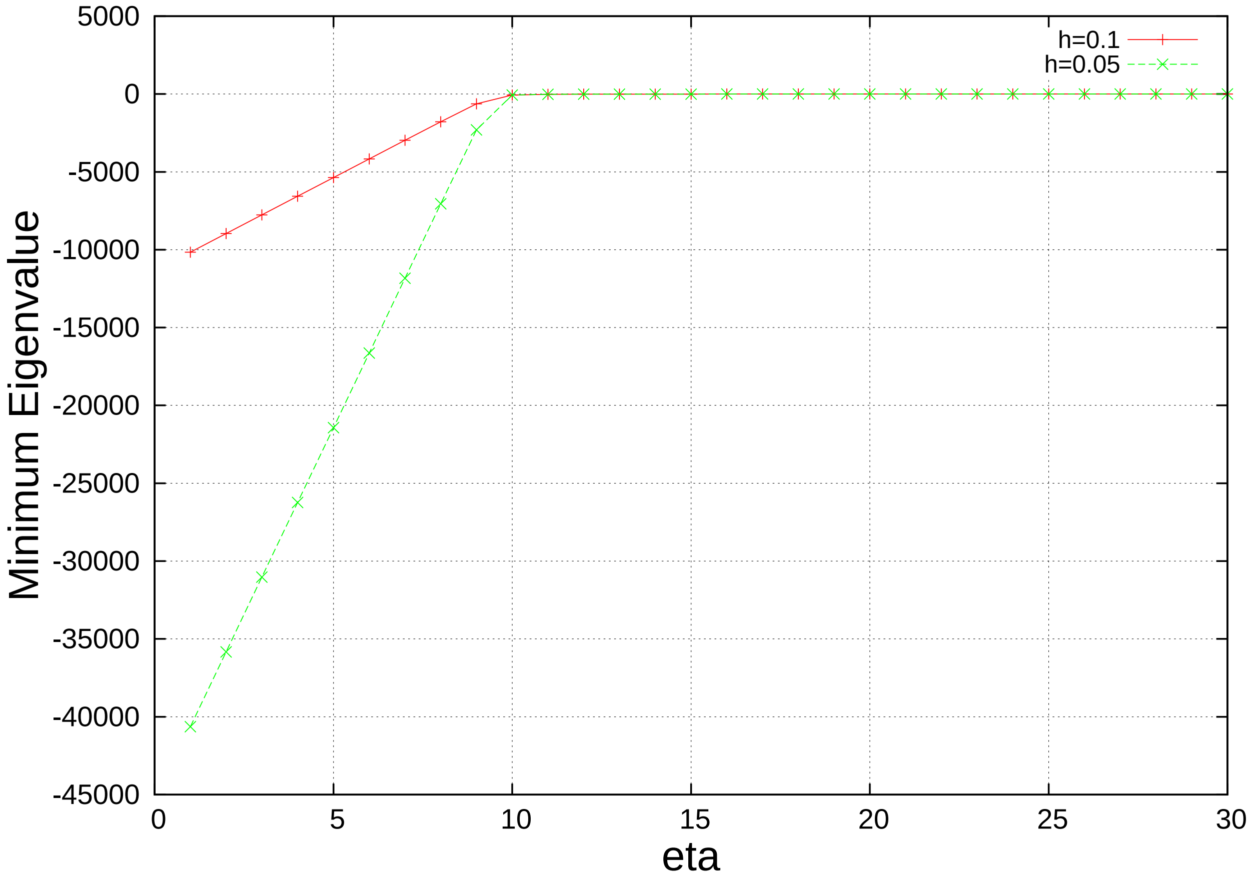
Exact Lower Bound of $\eta^h_{\rm LB}$

To numerically seek the exact lower bound $\eta^h_{\rm LB}$, we compute the minimum eigenvalue of the matrix B^h_η defined by

$$\left(B^h_\eta \vec{u}^h,\, \vec{v}^h\right)_{\mathbb{R}^n} = b^h_\eta(\underline{\boldsymbol{u}}^h,\, \underline{\boldsymbol{v}}^h) \quad \forall \underline{\boldsymbol{u}}^h,\, \underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}^h,$$

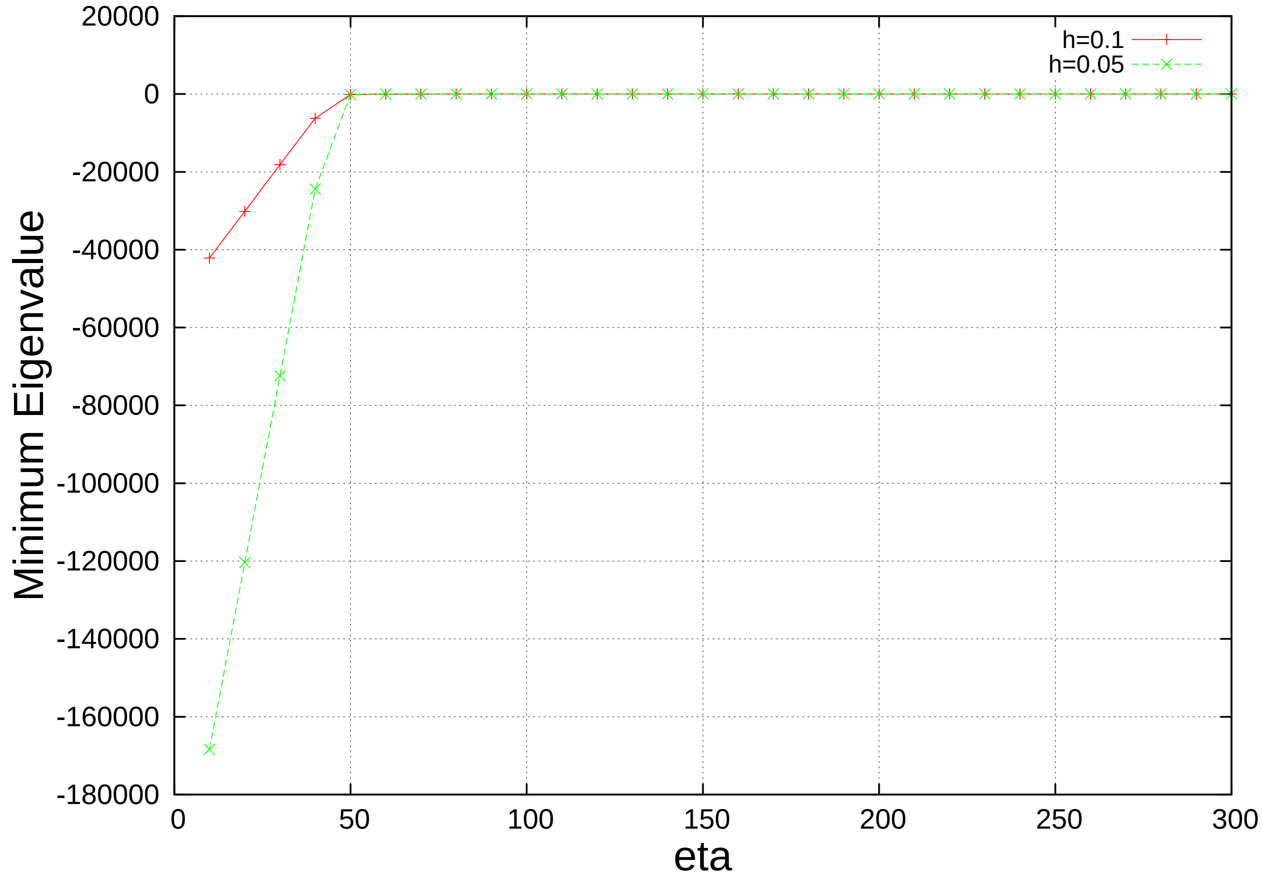
where we identify $\underline{\boldsymbol{V}}^h$ with $\mathbb{R}^n$ ($n := \dim \underline{\boldsymbol{V}}^h$), and correspondingly $\underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}^h$ with $\vec{v}^h \in \mathbb{R}^n$.

Minimum eigenvalue of B_η^h



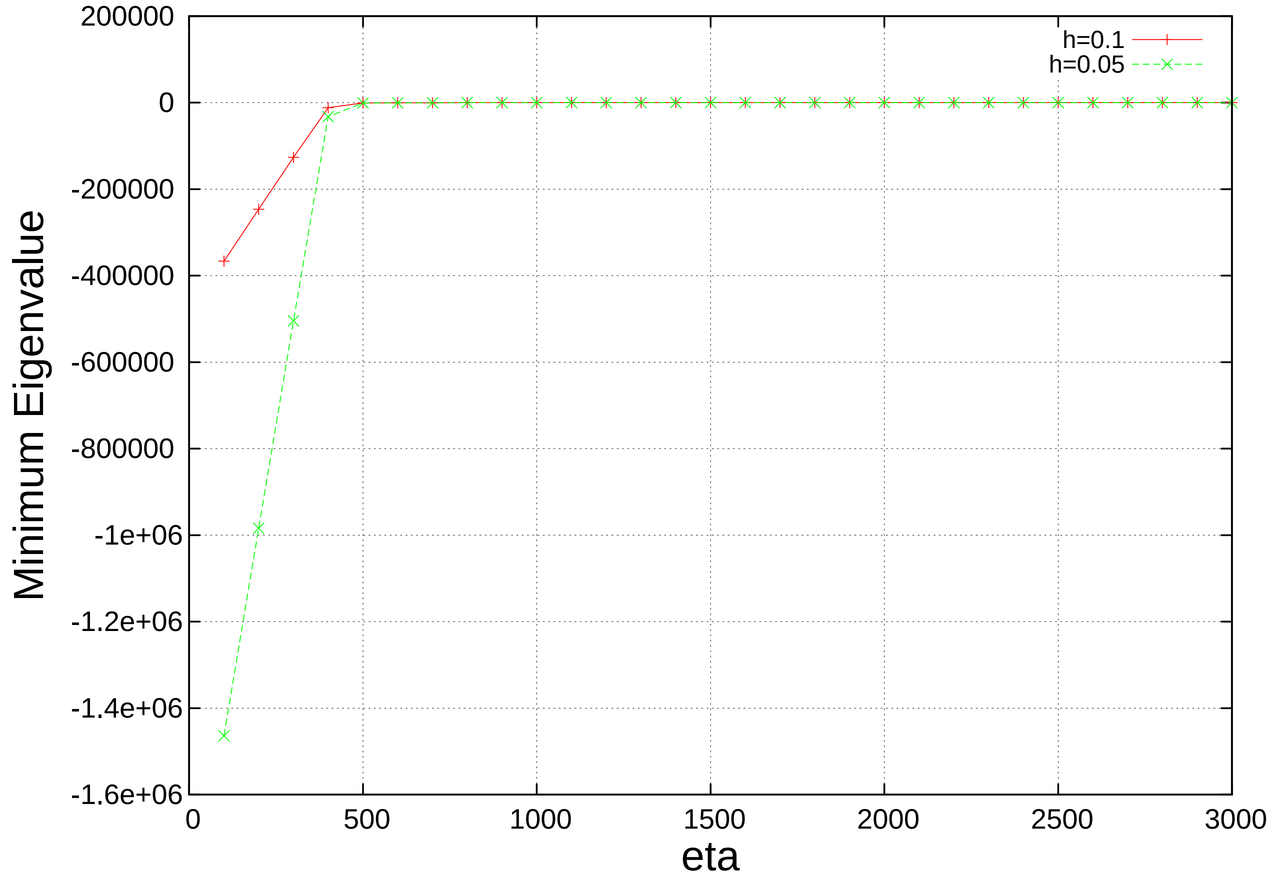
$$\lambda = 1$$

Minimum eigenvalue of B_η^h



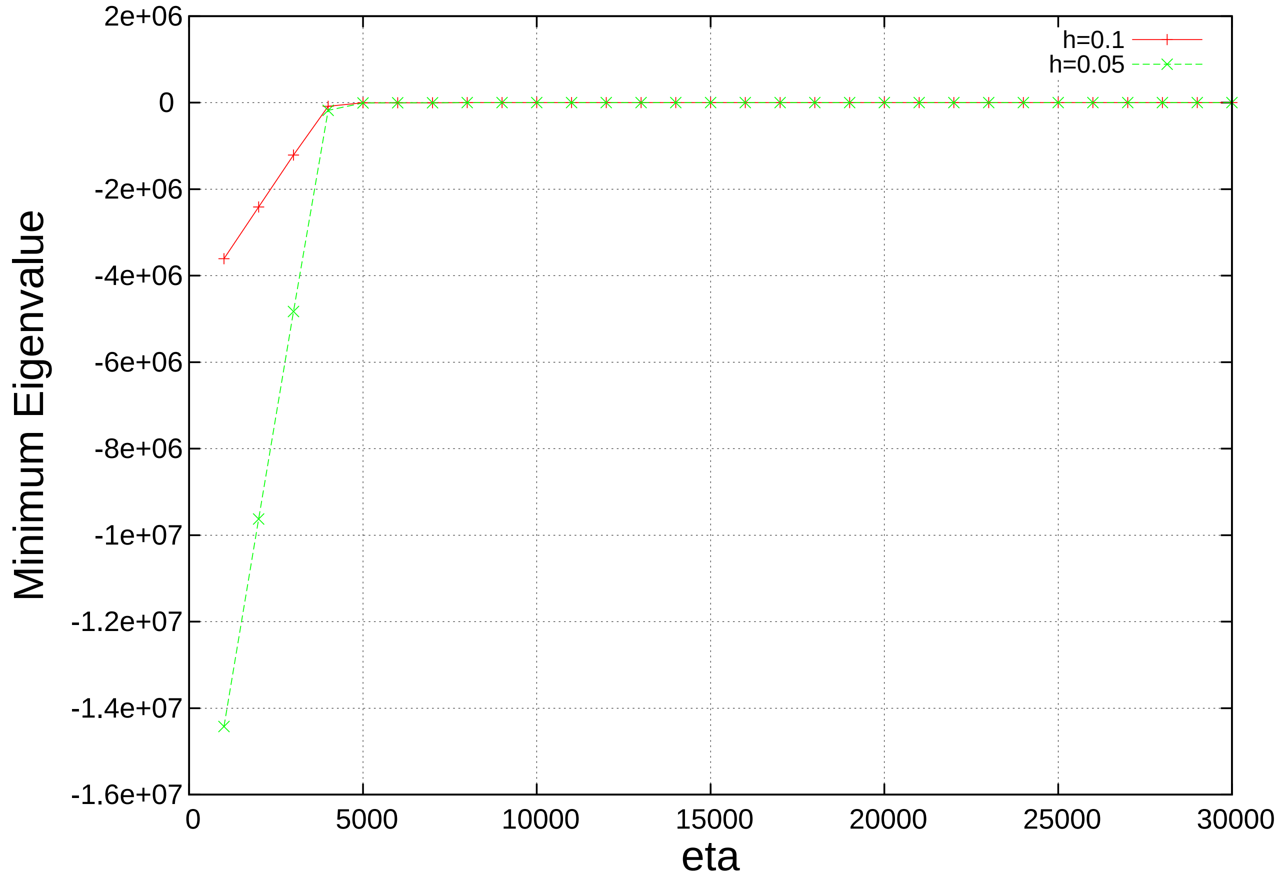
$\lambda = 10$

Minimum eigenvalue of B_η^h



$$\lambda = 100$$

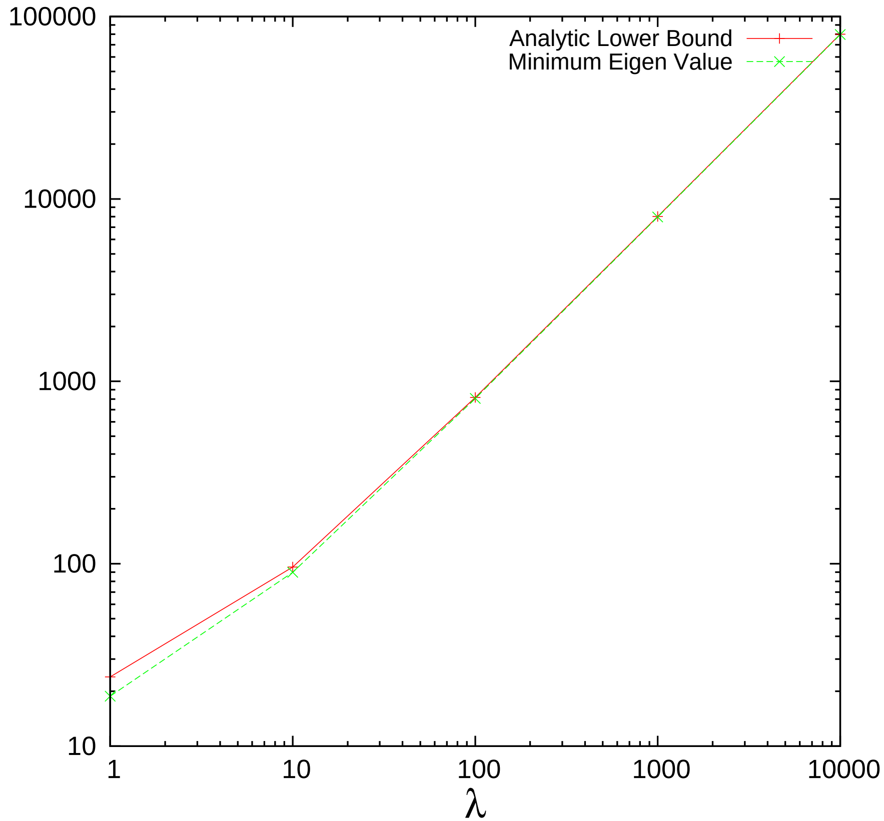
Minimum eigenvalue of B_η^h



$$\lambda = 1000$$

Comparison between η_0 and η_{LB}^h

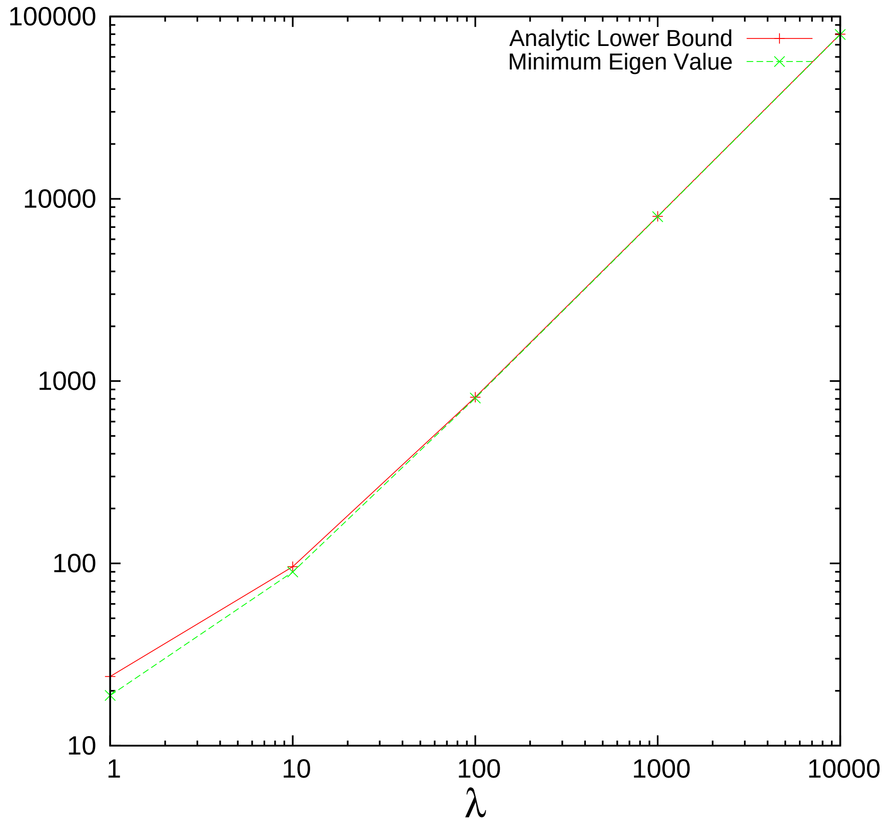
We plot η_0 and η_{LB}^h for $\lambda = 10^i$ $(i = 0, 1, \ldots, 4)$ in the figures below, where the red line displays η_0 and the green one η_{LB}^h.



$h = 1/10$

Comparison between η_0 and η_{LB}^h

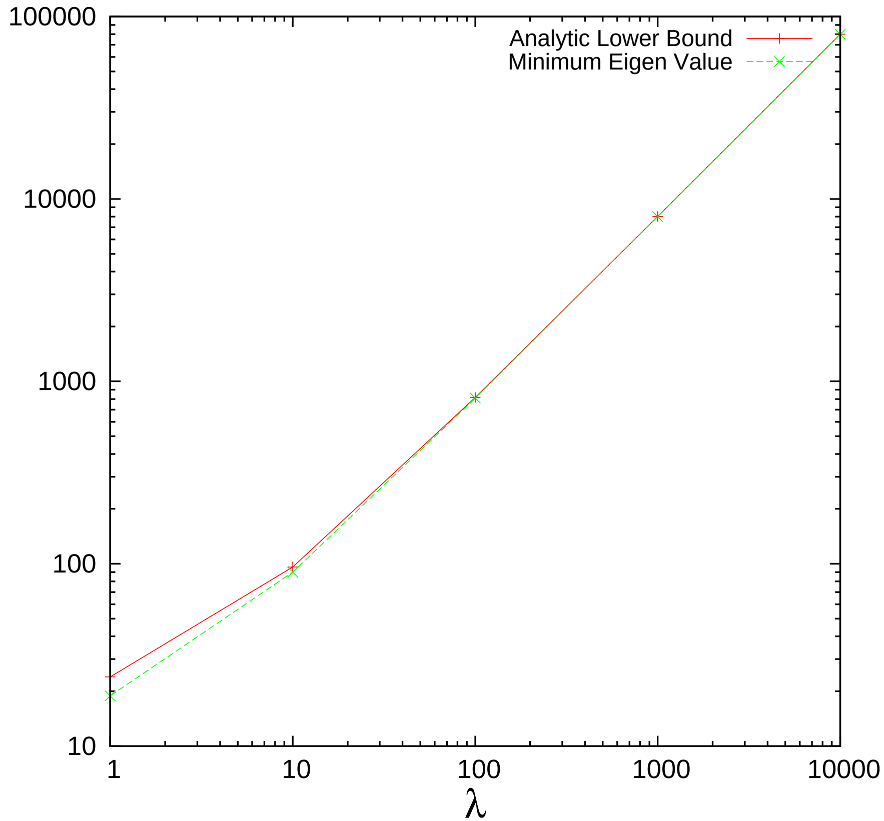
We plot η_0 and η_{LB}^h for $\lambda = 10^i$ $(i = 0, 1, \ldots, 4)$ in the figures below, where the red line displays η_0 and the green one η_{LB}^h.



$h = 1/20$

Comparison between η_0 and η_{LB}^h

We plot η_0 and η_{LB}^h for $\lambda = 10^i$ $(i = 0, 1, \ldots, 4)$ in the figures below, where the red line displays η_0 and the green one η_{LB}^h.



$h = 1/40$

Comparison between η_0 and η_{LB}^h

λ	10^0	10^1	10^2	10^3	10^4
$\eta_0 = 8(\lambda + 2\mu)$	24.00	96.00	816.0	8016	80016
$\eta_{\mathrm{LB}}^h\ (h = 1/10)$	18.78	89.69	805.6	7969	79625
$\eta_{\mathrm{LB}}^h\ (h = 1/20)$	18.86	89.87	807.2	7976	79703
$\eta_{\mathrm{LB}}^h\ (h = 1/40)$	18.89	89.96	808.8	7992	79859

- We can observe that η_0 is a good estimation of η_{LB}^h.
- Hence we MUST take $\eta = O(\lambda)$ in DG-woL as $\lambda \longrightarrow \infty$.

Comparison between η_0 and η^h_{LB}

- Here we note that the solution of DG(-wL or -woL) $\underline{\boldsymbol{u}}^h_\eta$ converges to the solution of the conforming FEM $\underline{u}^h_{\mathrm{FEM}}$ as $\eta \longrightarrow \infty$, that is,

$$\|\underline{\boldsymbol{u}}^h_\eta - \underline{\boldsymbol{u}}^h_{\mathrm{FEM}}\|_{\underline{\boldsymbol{V}}^h} = O(\eta^{-1/2}) \quad (\eta \longrightarrow \infty).$$

- This suggests that if we take $\eta = O(\lambda)$ as $\lambda \longrightarrow \infty$, then locking phenomena may occur, because P_1 conforming FEM causes locking phenemena.
- We will show this fact theoretically and numerically in what follows.

Theoretical Analysis

An a priori error estimate for DG-wL

Theorem 1 *Assume that* $\lambda > 0$ *and* $\eta \in [\eta_1, \eta_2]$ *with* $0 < \eta_1 < \eta_2$. *Let* $\underline{u} \in H_0^1(\Omega)^2$ *be the solution of the plane strain problem. Assume that* $\underline{u} \in \underline{H}^2(\Omega)$. *Further let* $\underline{\hat{u}} := \underline{u}|_{\Gamma^h}$. *Let* $\underline{\boldsymbol{u}}^h \in \underline{\boldsymbol{V}}^h$ *be the solution of DG-wL. Then we have*

$$\|\underline{\boldsymbol{u}} - \underline{\boldsymbol{u}}^h\|_{\underline{\boldsymbol{V}}^h} \leq Ch\|\underline{u}\|_{2,\Omega},$$

where C *is a positive constant independent of* $\lambda > 0$, η, h, *and* $\underline{u}$.

Sketch of Proof of Theorem 1

- This can be proved by a well-known method, which is also used in Hansbo–Larson (2002), Wihler (2004), Di Pietro–Nicaise (2013), and so on.
- That is, we reformulate the elasticity problem as a Stokes problem with nonzero divergence constrain, and establish a uniform inf-sup condition.
- The uniform inf-sup condition can be established by the method of proof due to Egger-Waluga (2013).

Locking Ratio

We define the so-called locking ratio due to Babuška–Suri (1992).

- For $\lambda > 0$, we define a solution space:

$$B^\lambda := \Big\{ \underline{v} \in \underline{H}^2(\Omega) \cap \underline{H}^1_0(\Omega) \mid \|\underline{v}\|_{H^2(\Omega)} + \lambda \|\operatorname{div} \underline{v}\|_{H^1(\Omega)} \le 1 \Big\}.$$

- For every $\underline{u} \in B^\lambda$ and for every $\lambda > 0$, let $\underline{\boldsymbol{u}}^h_\lambda \in \underline{\boldsymbol{V}}^h$ satisfy

$$a^h_\eta(\underline{\boldsymbol{u}}^h_\lambda, \underline{\boldsymbol{v}}^h) = a^h_\eta(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}^h) \quad \forall \underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}^h,$$

where $\underline{\boldsymbol{u}} := \{\underline{u}, \underline{u}|_{\Gamma^h}\}$.

Locking Ratio

We define the locking ratio $L(\lambda,\, h)$ for $\lambda > 0$ and $h \in (0,\, \bar{h}]$,

$$L(\lambda,\, h) := \frac{\sup_{\underline{u} \in B^\lambda} \|\underline{\boldsymbol{u}} - \underline{\boldsymbol{u}}_\lambda^h\|_{\underline{\boldsymbol{V}}^h}}{\sup_{\underline{u} \in B^\lambda} \inf_{\underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}^h} \|\underline{\boldsymbol{u}} - \underline{\boldsymbol{v}}^h\|_{\underline{\boldsymbol{V}}^h}}.$$

Now there exist positive constants C_1 and C_2 such that

$$C_1 h \le \sup_{\underline{u} \in B^\lambda} \inf_{\underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}^h} \|\underline{\boldsymbol{u}} - \underline{\boldsymbol{v}}^h\|_{\underline{\boldsymbol{V}}^h} \le C_2 h \quad \forall h \in (0,\, \bar{h}].$$

This implies that we may redefine the locking ratio as follows:

$$L(\lambda,\, h) := \frac{\sup_{\underline{u} \in B^\lambda} \|\underline{\boldsymbol{u}} - \underline{\boldsymbol{u}}_\lambda^h\|_{\underline{\boldsymbol{V}}^h}}{h} \quad \text{(cf. [21, 7]).}$$

DG-wL is locking free.

DG-wL is locking free with respect to the solution set B^λ and the norm $\|\cdot\|_{\underline{\boldsymbol{V}}^h}$ in the sense of Babuška-Suri, i.e.,

$$\lim_{h\longrightarrow+0}\sup\sup_{\lambda>0} L(\lambda,\,h) < \infty.$$

Indeed, we see from the a priori error estimate in Theorem 1 that

$$\frac{\|\underline{\boldsymbol{u}} - \underline{\boldsymbol{u}}^h\|_{\underline{\boldsymbol{V}}^h}}{h} \le C\|\underline{u}\|_{2,\Omega} \le C,$$

where C is a positive constant independent of h and λ.

DG-woL shows locking of order h^{-1}

In DG-woL, we must take $\eta = O(\lambda)$. So we assume $\eta = c\lambda$, where c is a positive constant.
Let $\underline{\boldsymbol{u}}_\lambda^h \in \underline{\boldsymbol{V}}^h$ satisfy

$$b_{c\lambda}^h(\underline{\boldsymbol{u}}_\lambda^h, \underline{\boldsymbol{v}}^h) = b_{c\lambda}^h(\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}}^h) \quad \forall \underline{\boldsymbol{v}}^h \in \underline{\boldsymbol{V}}^h.$$

We now pose a hypothesis:

(L) $\quad \left\{ \underline{v}^h \in \underline{V}_c^h \mid \operatorname{div} \underline{v}^h = 0 \right\} = \{\underline{0}\} \quad \forall h \in (0, \bar{h}],$

where

$$\underline{V}_c^h := \underline{U}^h \cap \underline{H}_0^1(\Omega) \quad (P_1 \text{ conforming FE space}).$$

It is well-known that almost all triangulations satisfy (L) (see Mercier(1979), Boffi–Brezzi–Fortin(2013)).

DG-woL shows locking of order h^{-1}

Theorem 2 *Assume that a family of triangulations $\{\mathcal{T}^h\}_{0<h\leq\bar{h}}$ satisfies (L). DG-woL with $\eta = c\lambda\ (c>0)$ shows locking of order h^{-1} with respect to the solution set B^λ and the norm $\|\cdot\|_{\underline{\boldsymbol{V}}^h}$ in the sense of Babuška–Suri, that is,*

$$0 < \limsup_{h\longrightarrow+0}\left[h\sup_{\lambda>0} L(\lambda,\, h)\right] < +\infty.$$

Proof. This is established in a similar way to the way which Brenner–Scott(2008) used to prove that P_1 conforming FEM causes locking phenomena. ■

Numerical Examples

Test problem

We use the same test problem that we used at the start.

- Domain $\Omega := (0,\, 1) \times (0,\, 1)$.
- We fix Lamé parameter $\mu = 1$.
- We determine the exact solution $\underline{u}$ by

$$\begin{aligned} \psi(x) &:= x^2(x-1)^2, \\ \Psi(x_1,\, x_2) &:= -\frac{1}{2}\psi(x_1)\psi(x_2) \quad \text{(stream function)}, \\ \underline{u} &:= \operatorname{rot}\Psi. \end{aligned}$$

- The exact solution is independent of λ and satisfies $\operatorname{div}\boldsymbol{u} = 0$.
- This test problem is presented in Bercovier–Livne (1979) and Soon–Cockburn–Stolarski (2009).

Test problem

- We use 4 meshes which are obtained by dividing each side of Ω into $2^j \times 10$ $(j = 0, 1, \ldots, 3)$ equi-length line segments.

	meshes	η
DG-wL	unstructured	1
DG-woL	structured (FK type)	$\eta_0 \equiv 8(\lambda + 2\mu)$

Locking ratio

- Let $\underline{u}$ be the exact solution.
- We consider the following solution set:

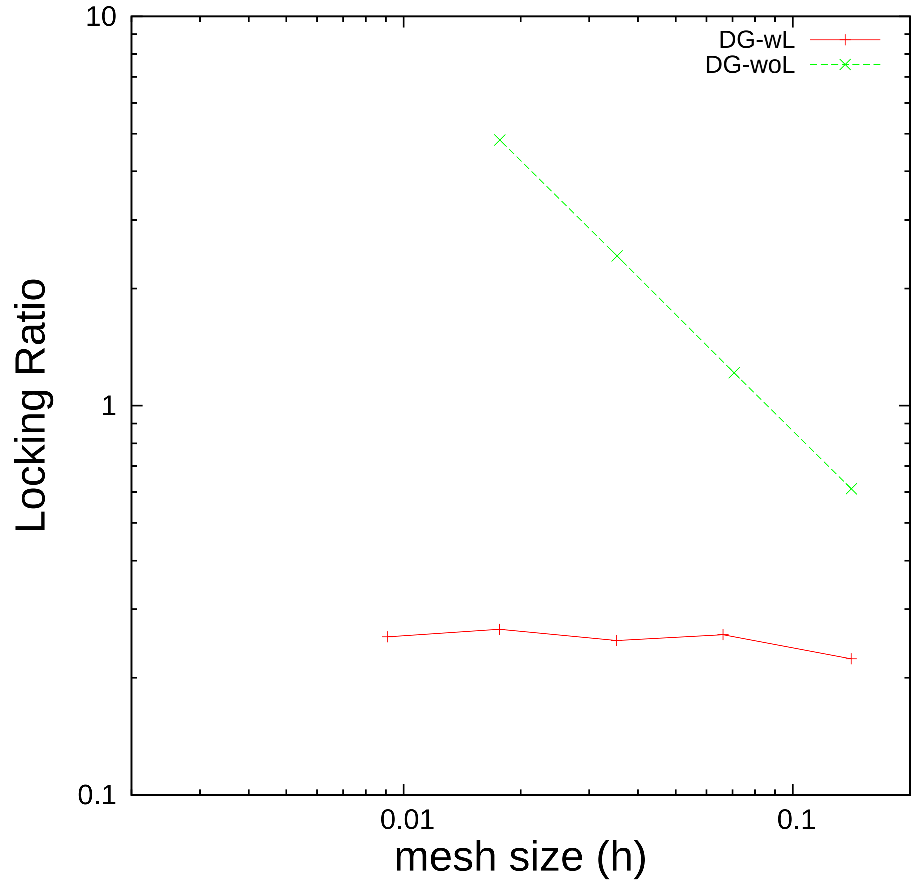
$$B^\lambda := \{\alpha \underline{u} \mid |\alpha| \leq 1\}.$$

- Let $L(\lambda, h)$ be the locking ratio with respect to the solution set B^λ and the norm $\|\cdot\|_{\underline{\boldsymbol{V}}^h}$.
- As an approximation of $\sup_{\lambda>0} L(\lambda, h)$, we compute

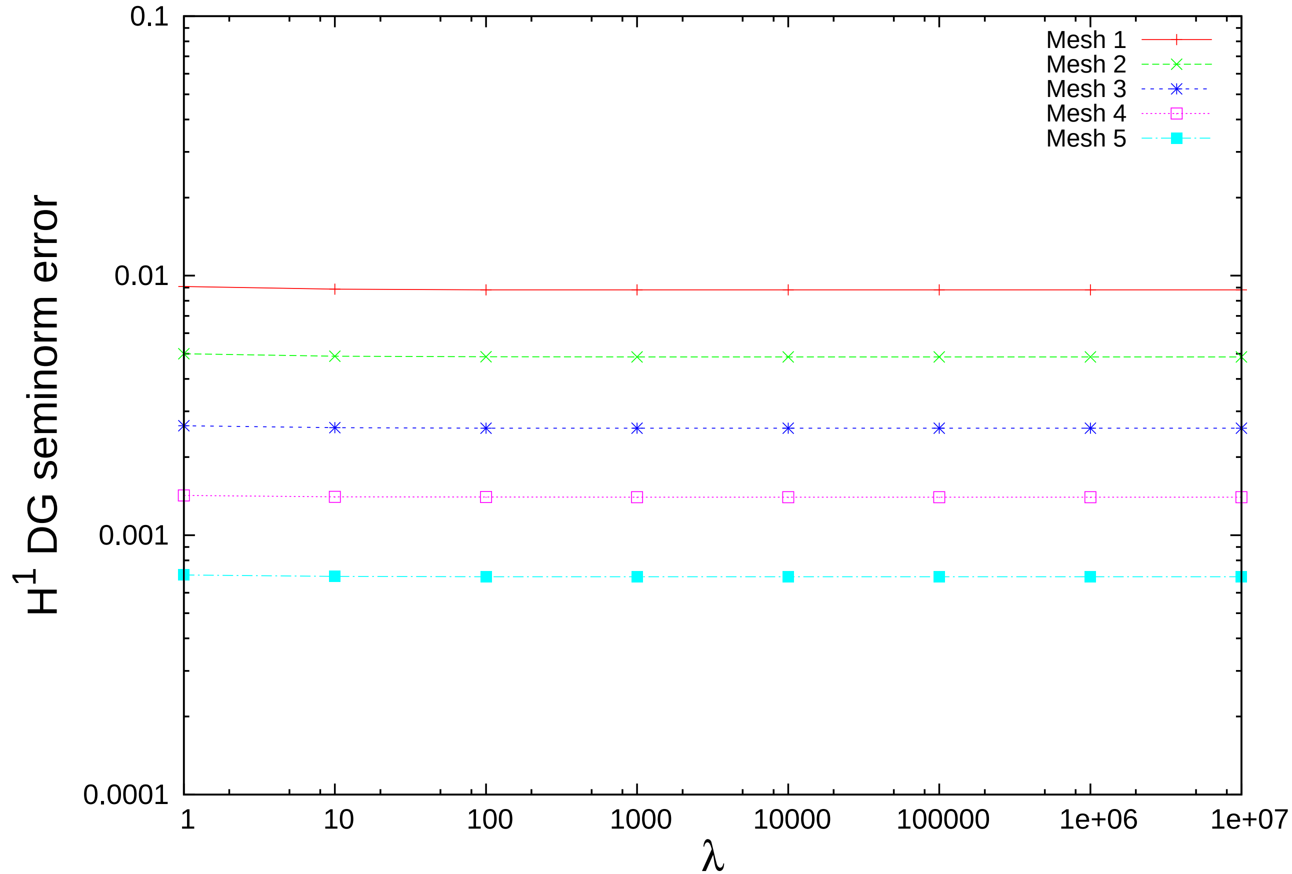
$$\max_{\lambda \in \Lambda} L(\lambda, h) \quad \left(\Lambda := \{10^j \mid j = 0, 1, \ldots, 12\}\right).$$

- We plot these values for DG-wL and DG-woL in the following figure.

Locking ratio

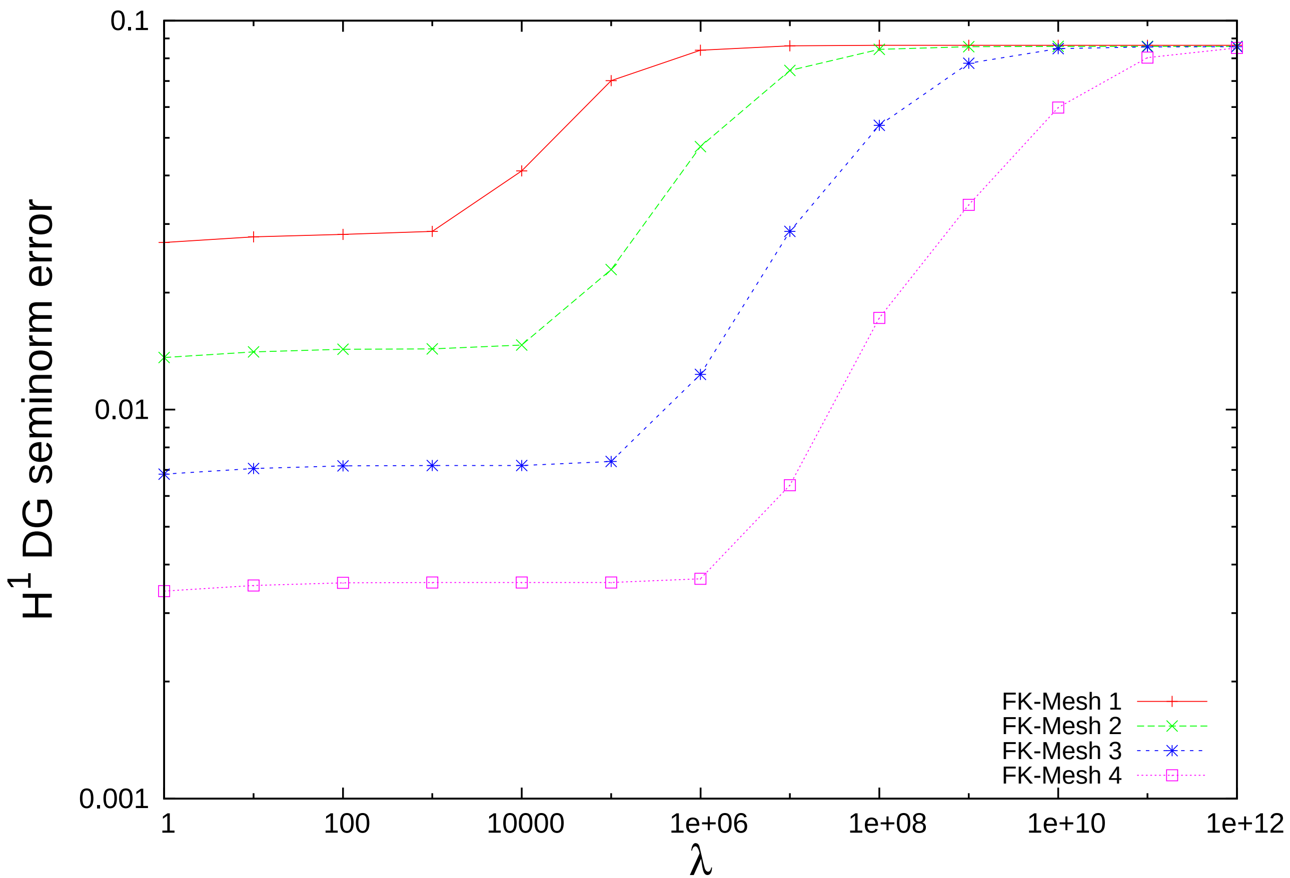


Error vs. $\lambda \in \Lambda$



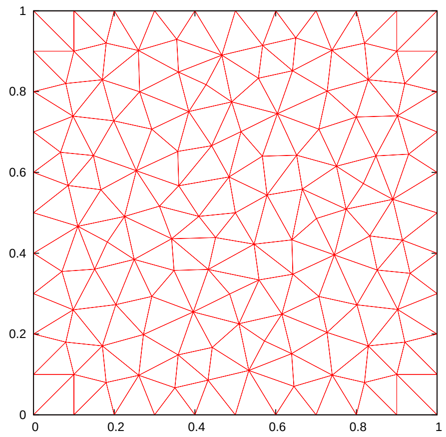
DG-wL

Error vs. $\lambda \in \Lambda$

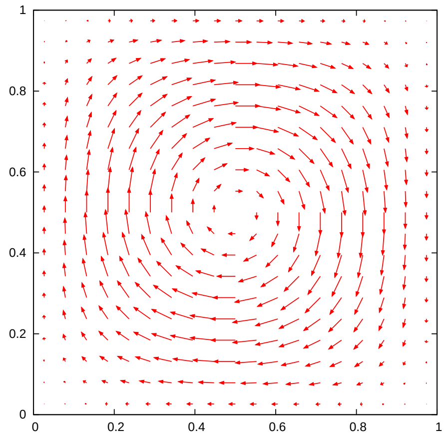


DG-woL

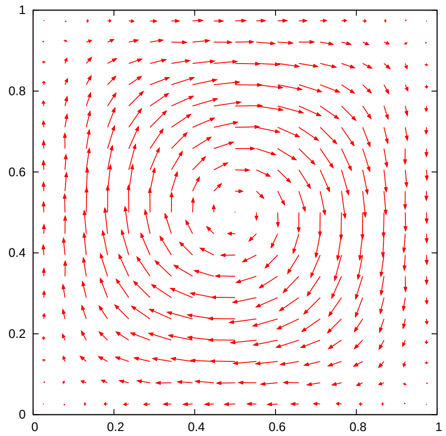
Vector fields by DG-wL $(\lambda = 10^7)$



$h = 1/10$

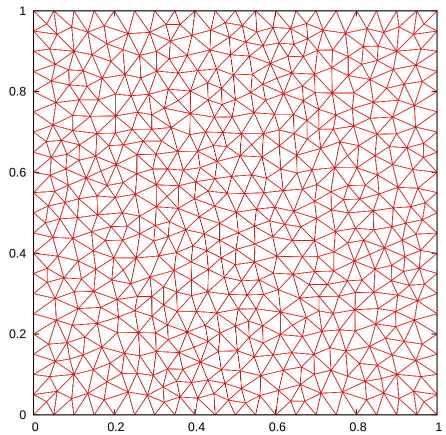


Exact sol. $\underline{u}$

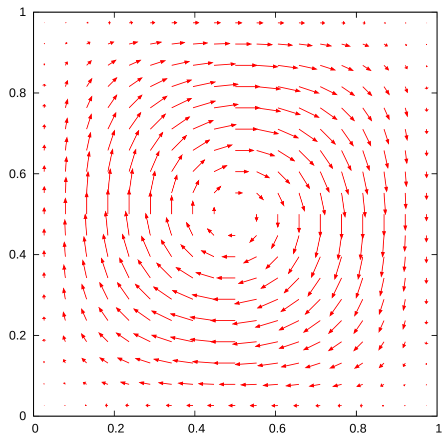


DG-wL

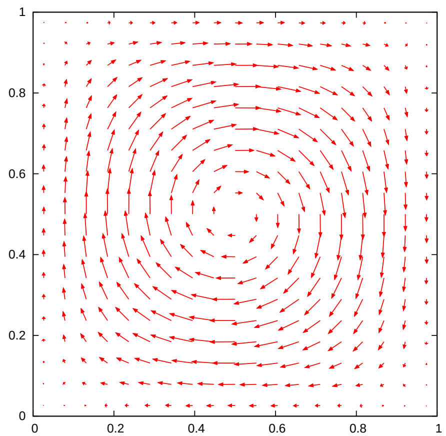
Vector fields by DG-wL $(\lambda = 10^7)$



$h = 1/20$

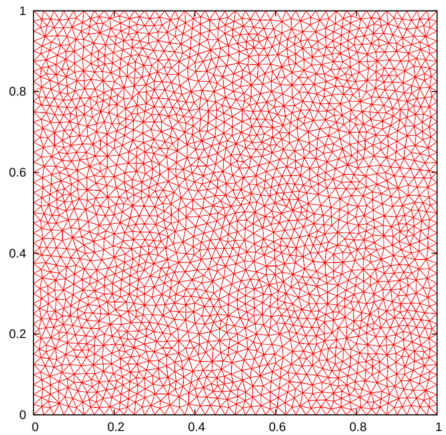


Exact sol. $\underline{u}$

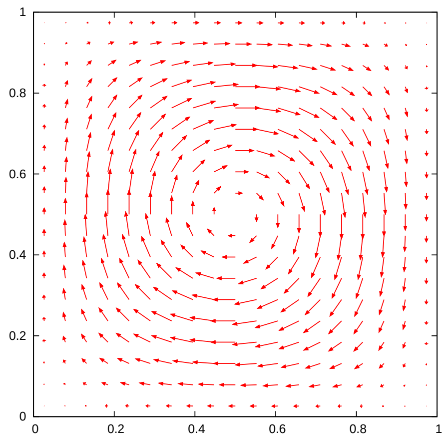


DG-wL

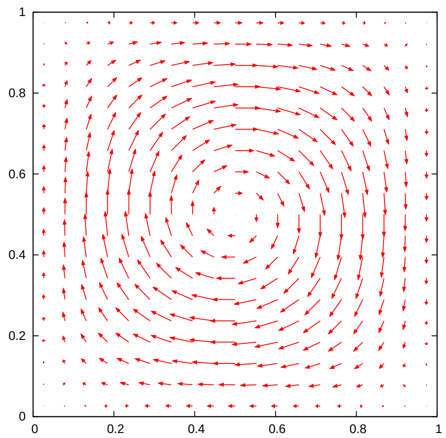
Vector fields by DG-wL $(\lambda = 10^7)$



$h = 1/40$

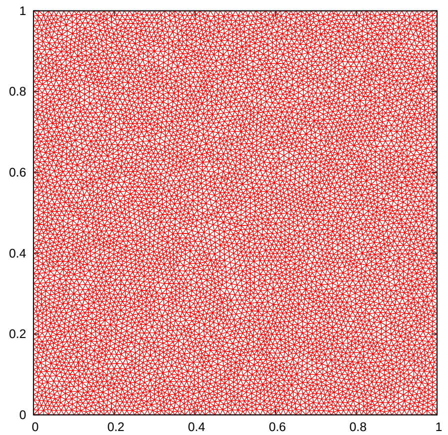


Exact sol. $\underline{u}$

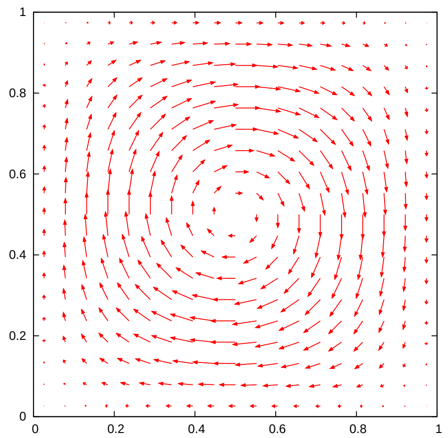


DG-wL

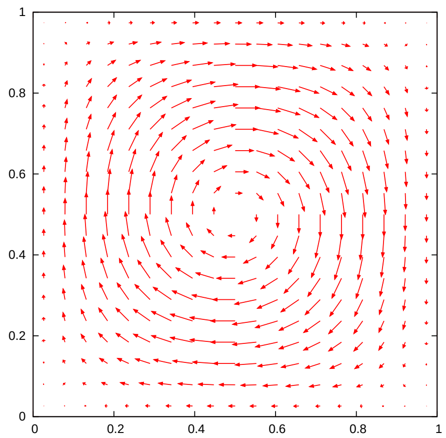
Vector fields by DG-wL $(\lambda = 10^7)$



$h = 1/80$

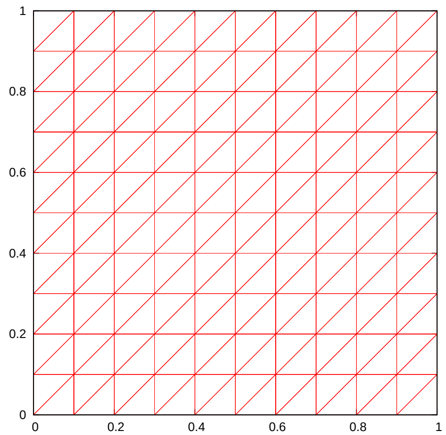


Exact sol. $\underline{u}$

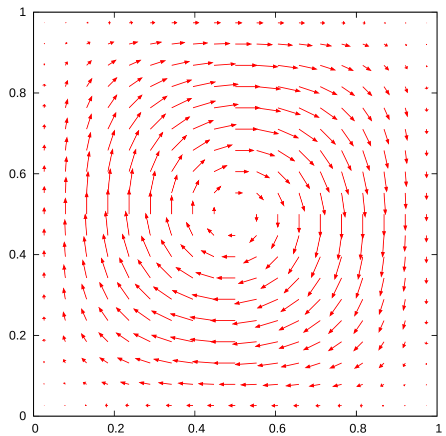


DG-wL

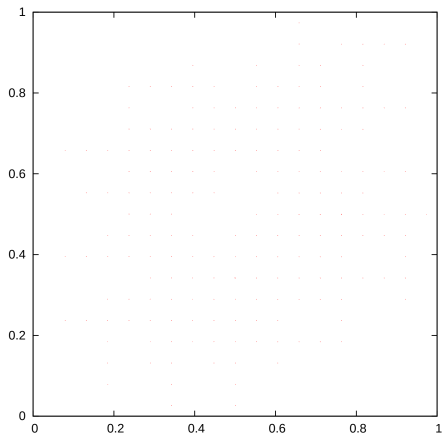
Vector fields by DG-woL $(\lambda = 10^7)$



$h = 1/10$

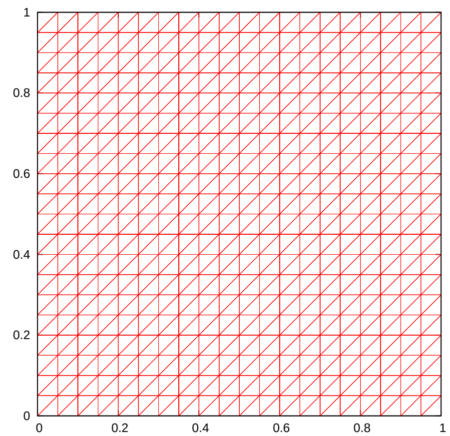


Exact sol. $\underline{u}$

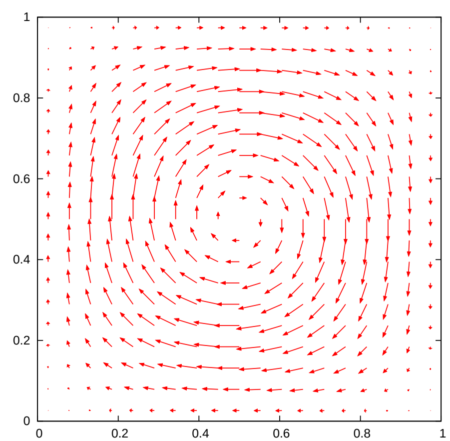


DG-woL

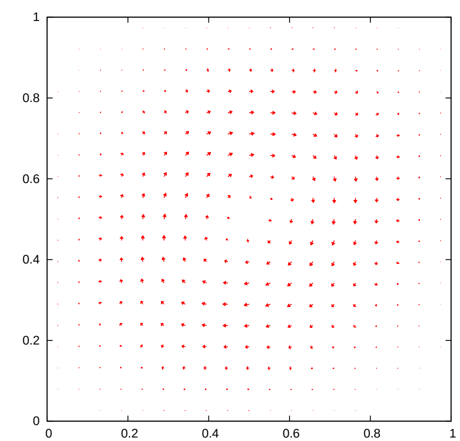
Vector fields by DG-woL $(\lambda = 10^7)$



$h = 1/20$

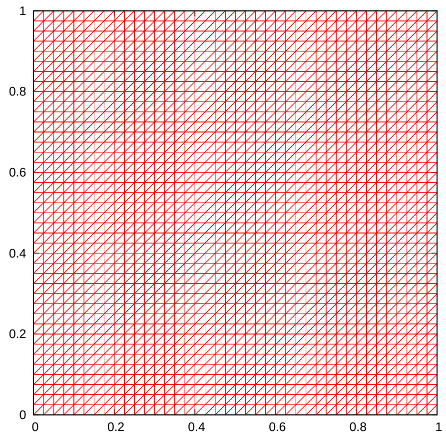


Exact sol. $\underline{u}$

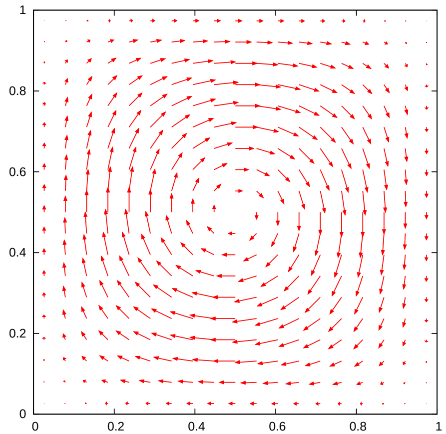


DG-woL

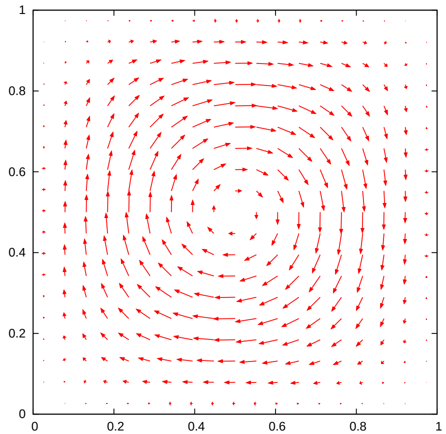
Vector fields by DG-woL $(\lambda = 10^7)$



$h = 1/40$

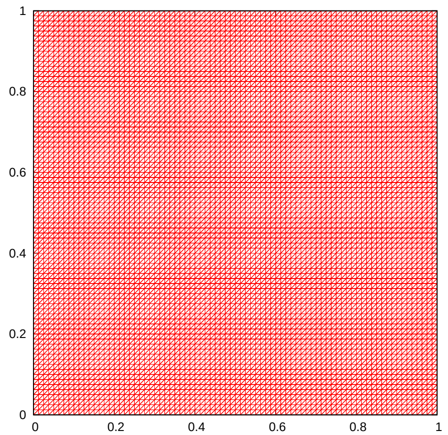


Exact sol. $\underline{u}$

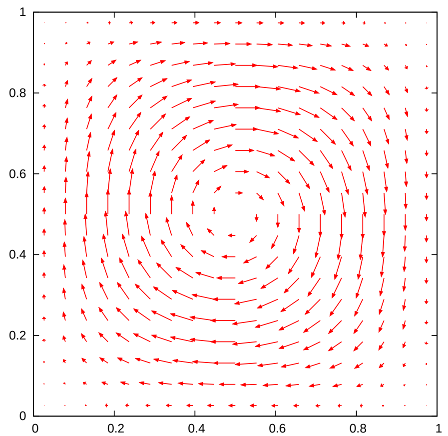


DG-woL

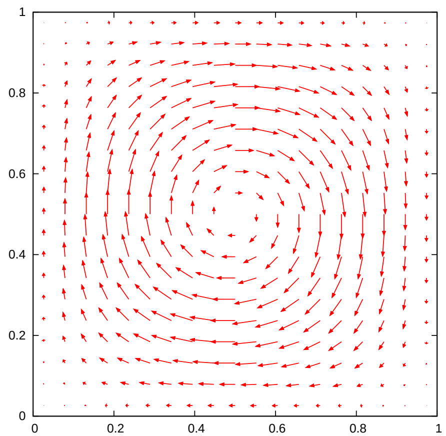
Vector fields by DG-woL $(\lambda = 10^7)$



$h = 1/80$



Exact sol. $\underline{u}$



DG-woL

Conclusion

Conclusion

- DG-wL prevents volume locking phenomena. Because we can choose a small η in DG-wL.
- On the other hand, when we use DG-woL, we have to choose $\eta = O(\lambda)$ $(\lambda \longrightarrow \infty)$. This choice causes volume locking phenomena.
- We conclude that the lifting term is important for avoiding the volume locking in our Hybrid DGFEM formulation.

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