A broken P1 nonconforming finite element methods for the elasticity problems with the interface

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Abstract

We propose new schemes for solving linear elasticity problems consisting of heterogeneous materials. Recently, immersed finite element methods (IFEM) are developed for partial differential equation with discontinuous coefficients. IFEM use uniform grids allowing the interface to cut through the elements. To use uniform grids, we develop P1-nonconforming based on IFEM functions which satisfy Laplace-Young conditions along the interface. We add stabilizing terms and consistency terms to the bilinear form to improve the results. Optimal rates of convergence are achieved in the numerical results.

Introduction

Let Ω be a connected, convex polygonal domain in $\mathbb{R}^2$ which is divided into two subdomains Ω^+ and Ω^- by a C^2 interface $\Gamma = \partial\Omega^+ \cap \partial\Omega^-$. We assume the subdomains Ω^+ and Ω^- are occupied by two different elastic materials. Then the displacement $\mathbf{u} = (u_1, u_2)$ of the elastic body under an external force satisfies the Navier-Lamé equation as follows.

$$\begin{aligned} -\mathrm{div}\boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{f} \ \text{ in } \Omega^s,\ (s = +, -) && (1)\\ [\mathbf{u}]_\Gamma &= 0, && (2)\\ [\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n}]_\Gamma &= 0, && (3)\\ \mathbf{u} &= 0 \ \text{ on } \partial\Omega, && (4)\end{aligned}$$

where

$$\boldsymbol{\sigma}(\mathbf{u}) = 2\mu\boldsymbol{\epsilon}(\mathbf{u}) + \lambda\, tr(\boldsymbol{\epsilon}(\mathbf{u}))\boldsymbol{\delta},\ \ \boldsymbol{\epsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$$

are the stress tensor and the strain tensor respectively, $\mathbf{n}$ is outward unit normal vector, $\boldsymbol{\delta}$ is the identity tensor, and $\mathbf{f} \in (L^2(\Omega))^2$ is the external force. The bracket $[\cdot]$ means the jump across the interface. Here

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}$$

are the Lame constants, satisfying $0 < \mu_1 < \mu < \mu_2$ and $0 < \lambda < \infty$, and E is the Young's modulus and ν is the Poisson ratio. When the parameter $\lambda \to \infty$, this equation describes the behavior of nearly incompressible material. Since the material properties are different in each region, we set the Lamé constants $\mu = \mu^s, \lambda = \lambda^s$ on Ω^s for $s = +, -$.

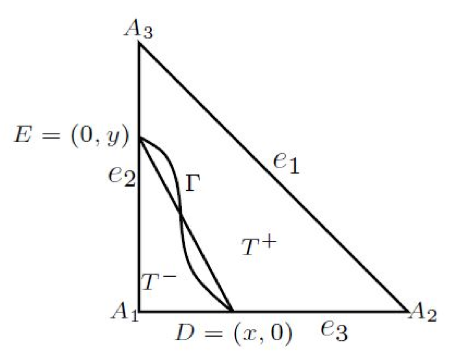
IFEM scheme

IFEM basis

The main idea of the IFEM is to use two pieces of linear shape functions on an interface element to satisfy the interface condition. In this case, we set, for $i = 1, 2, \cdots, 6$,

$$\hat{\boldsymbol{\phi}}_i(x,y) = \begin{cases} \hat{\phi}_i^+(x,y) = \begin{pmatrix}\hat{\phi}_{i1}^+\\ \hat{\phi}_{i2}^+\end{pmatrix} = \begin{pmatrix} a_1^+ + b_1^+x + c_1^+y\\ a_2^+ + b_2^+x + c_2^+y\end{pmatrix}, & (x,y) \in T^+\\ \hat{\phi}_i^-(x,y) = \begin{pmatrix}\hat{\phi}_{i1}^-\\ \hat{\phi}_{i2}^-\end{pmatrix} = \begin{pmatrix} a_1^- + b_1^-x + c_1^-y\\ a_2^- + b_2^-x + c_2^-y\end{pmatrix}, & (x,y) \in T^-\end{cases}$$

and require these functions satisfy the nodal value conditions(edge average), continuity, and jump conditions along the interface:



$$\begin{aligned} \overline{\hat{\phi}_{i1}|e_j} &= \delta_{ij},\ j = 1,2,3\\ \overline{\hat{\phi}_{i2}|e_j} &= \delta_{(i-3)j},\ j = 1,2,3\\ [\hat{\boldsymbol{\phi}}_i(D)] &= 0,\\ [\hat{\boldsymbol{\phi}}_i(E)] &= 0,\\ \left[\boldsymbol{\sigma}(\hat{\boldsymbol{\phi}}_i)\cdot\mathbf{n}\right]_{\overline{DE}} &= 0.\end{aligned}$$

Figure 1: A typical interface triangle

It is shown in [1] that these twelve conditions uniquely determine the basis functions $\hat{\boldsymbol{\phi}}_i$, $i = 1, \cdots, 6$. We denote by $\widehat{\mathbf{N}}_h(T)$ the space of functions generated by $\hat{\boldsymbol{\phi}}_i, i = 1, \cdots, 6$ constructed above. Using this IFEM basis, we define the global *IFEM space* $\widehat{\mathbf{N}}_h(\Omega)$ by

$$\widehat{\mathbf{N}}_h(\Omega) = \left\{ \begin{array}{l} \hat{\boldsymbol{\phi}} \in \widehat{\mathbf{N}}_h(T) \text{ if T is an interface element, and } \hat{\boldsymbol{\phi}} \in P_1(T) \text{ if T is not an interface element;}\\ \text{if } T_1 \text{ and } T_2 \text{ share an edge } e, \text{ then } \int_e \hat{\boldsymbol{\phi}}|_{\partial T_1} ds = \int_e \hat{\boldsymbol{\phi}}|_{\partial T_2} ds; \text{ and } \int_{\partial T \cap \partial\Omega} \hat{\boldsymbol{\phi}}\, ds = \mathbf{0}\end{array}\right\}.$$

Approximation property of $\widehat{\mathbf{N}}_h(\Omega)$

Proposition 1. *For any $\mathbf{u} \in (\widetilde{H}^2_\Gamma(\Omega))^2$, there exists a constant $C > 0$ such that for $m = 0, 1$*

$$\|\mathbf{u} - I_h\mathbf{u}\|_{m,h} + m \cdot \|\sqrt{\lambda}\mathrm{div}(\mathbf{u} - I_h\mathbf{u})\|_{L^2(\Omega)} \le Ch^{2-m}(\|\mathbf{u}\|_{\widetilde{H}^2(\Omega)} + m \cdot \sqrt{\lambda_M}\|\mathrm{div}\mathbf{u}\|_{\widetilde{H}^1(\Omega)}),$$

and

$$\|\mathbf{u} - I_h\mathbf{u}\|_{m,h} \le Ch^{2-m}\|\mathbf{u}\|_{\widetilde{H}^2(\Omega)}.$$

Scheme and error estimate

We propose new IFEM schemes for (1)-(4). Find $\mathbf{u}_h \in \widehat{\mathbf{N}}_h(\Omega)$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \widehat{\mathbf{N}}_h(\Omega), \tag{5}$$

where

$$\begin{aligned} a_h(\mathbf{u},\mathbf{v}) := &\sum_{T\in\mathcal{T}_h}\left(\int_T 2\mu\boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v})dx + \int_T \lambda \mathrm{div}\mathbf{u}\,\mathrm{div}\mathbf{v}\,dx\right) + \sum_{e\in\mathcal{E}}\int_e \frac{\tau}{h}[\mathbf{u}][\mathbf{v}]ds\\ &-\sum_{e\in\mathcal{E}}\int_e \{\boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n}\}\cdot[\mathbf{v}]ds - \epsilon\sum_{e\in\mathcal{E}}\int_e\{\boldsymbol{\sigma}(\mathbf{v})\cdot\mathbf{n}\}\cdot[\mathbf{u}]ds,\ \forall\mathbf{u},\mathbf{v}\in\mathbf{H}_h(\Omega).\end{aligned}$$

Since the nonconforming basis does not hold Korn's inequality, we have to add the stabilized terms (the third term in scheme) in order to converge robustly. And the last two terms of the above equation are not necessary in standard case. But we can avoid consistency estimate by adding these terms. The variations of the scheme are motivated to IIPG, SIPG, and NIPG of DG methods ($\epsilon = 0, \epsilon = 1$, and $\epsilon = -1$, respectively). At last, we introduce the following mesh dependent energy-like norms.

$$\begin{aligned}\|\mathbf{v}\|^2_{a_h} := &\sum_{T\in\mathcal{T}_h}\int_T 2\mu\epsilon(\mathbf{v}) : \epsilon(\mathbf{v})dx + \int_T \lambda|\mathrm{div}\mathbf{v}|^2dx\\ &+\sum_{e\in\mathcal{E}}\left(\int_e h|\{\boldsymbol{\sigma}(\mathbf{v})\cdot\mathbf{n}\}|^2ds + \int_e\frac{\tau}{h}[\mathbf{v}]^2ds\right).\end{aligned}$$

Theorem 2. *Let $\mathbf{u}$ (resp. $\mathbf{u}_h$) be the solution of (1)(resp. (5)). Then we have*

$$\|\mathbf{u} - \mathbf{u}_h\|_{a_h} \le Ch\left(\|\mathbf{u}\|_{\widetilde{H}^2(\Omega)} + \sqrt{\lambda_M}\|\mathrm{div}\mathbf{u}\|_{\widetilde{H}^1(\Omega)}\right).$$

Numerical Results

The domain is $\Omega = (-1,1)\times(-1,1)$. The interface is the zero set of $L(x,y) = x^2 + y^2 - r_0^2$. Let $\Omega^+ = \Omega \cap \{(x,y)|L(x,y) > 0\}$, $\Omega^- = \Omega\cap\{(x,y)|L(x,y) < 0\}$. The exact solution is chosen as

$$\mathbf{u} = \left(\frac{1}{\mu}(x^2 + y^2 - r_0^2)x, \frac{1}{\mu}(x^2+y^2-r_0^2)y\right)$$

$1/h$	$\|\mathbf{u}-\mathbf{u}_h\|_0$	order	$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}$	order	$\|\mathrm{div}\mathbf{u}-\mathrm{div}\mathbf{u}_h\|_0$	order
8	7.799e-2		1.636e-0		1.059e-1	
16	2.418e-2	1.689	9.117e-1	0.844	5.514e-2	0.942
32	6.848e-2	1.820	4.521e-1	1.012	2.817e-2	0.969
64	1.860e-3	1.880	2.249e-1	1.007	1.417e-2	0.991
128	4.848e-4	1.940	1.112e-1	1.016	7.110e-3	0.995
256	1.237e-4	1.971	5.536e-2	1.006	3.563e-3	0.997

Table 1: $\mu^- = 1,\ \mu^+ = 10, \lambda = 1000\mu, r_0 = 0.6,$ nearly incompressible case

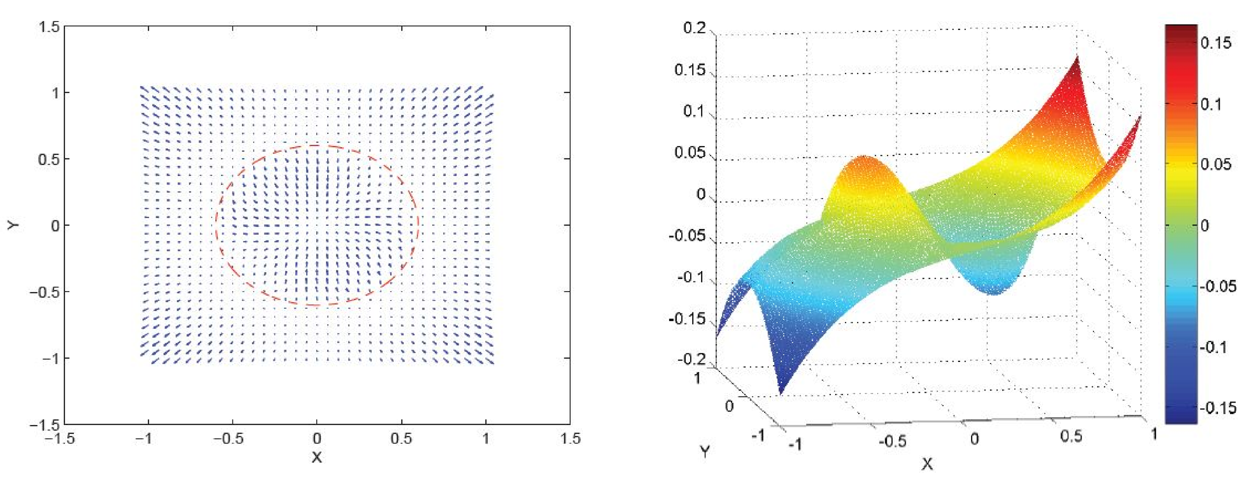


Figure 2: $\mathbf{u}$ and x-component of $\mathbf{u}$ figure about table 1

The domain is the same as above, and the interface is represented by $L(x,y) = \frac{x^2}{4} + y^2 - r_0^2 = 0$. The exact solution is chosen as

$$\mathbf{u} = \left(\frac{1}{\mu}(\frac{x^2}{4} + y^2 - r_0^2)x, \frac{1}{\mu}(\frac{x^2}{4}+y^2-r_0^2)y\right)$$

$1/h$	$\|\mathbf{u}-\mathbf{u}_h\|_0$	order	$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}$	order	$\|\mathrm{div}\mathbf{u}-\mathrm{div}\mathbf{u}_h\|_0$	order
8	1.791e-3		6.431e-2		5.906e-2	
16	4.543e-4	1.979	3.105e-2	1.050	2.902e-2	1.025
32	1.139e-4	1.996	1.568e-2	0.986	1.474e-2	0.977
64	2.785e-5	2.032	7.908e-3	0.987	7.452e-3	0.984
128	6.969e-6	1.999	3.970e-3	0.994	3.747e-3	0.992
256	1.740e-6	2.002	1.988e-3	0.998	1.878e-3	0.997

Table 2: $\mu^- = 1,\ \mu^+ = 10, \lambda = 5\mu, r_0 = 0.4,$ elliptical interface

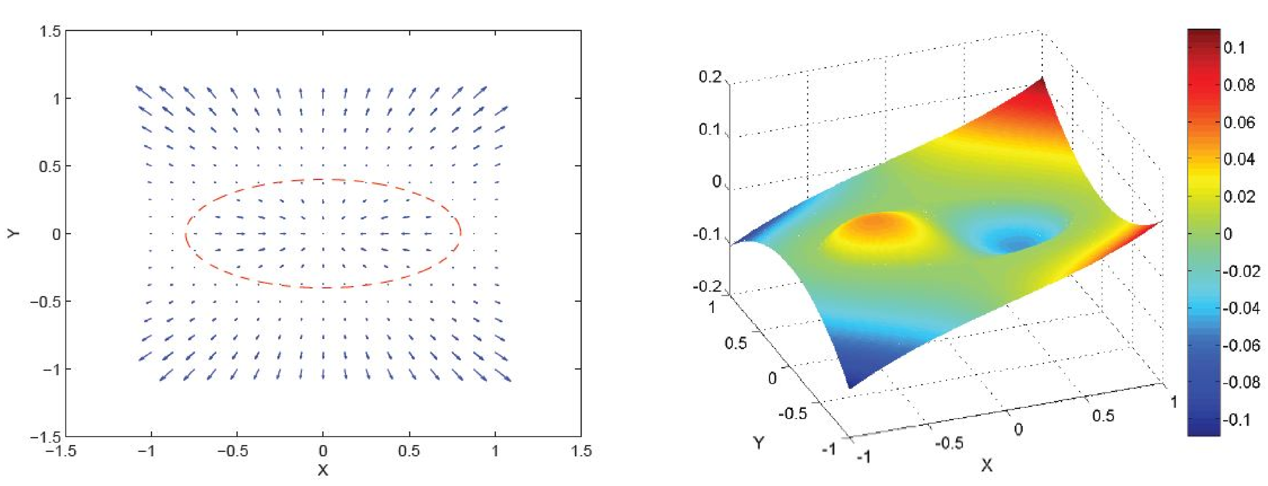


Figure 3: $\mathbf{u}$ and x-component of $\mathbf{u}$ figure about table 2

Conclusions

- We presented a numerical scheme using a uniform grid for the elasticity problem with an interface.
- This scheme is very useful in computing problems with moving interface since we use a grid independent of the interface.
- Numerical results show the optimal convergence rate in the case of various interface shapes and Lamé constants including nearly incompressible case.

References

[1] Do Y Kwak and Sangwon Jin. A stabilized p_1 immersed finite element method for the interface elasticity problems. *arXiv preprint arXiv:1408.4227*, 2014.

[2] Do Y Kwak, Kye T Wee, and Kwang S Chang. An analysis of a broken p_1-nonconforming finite element method for interface problems. *SIAM Journal on Numerical Analysis*, 48(6):2117–2134, 2010.